Hermitian Maaß Lift for General Level

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Saito-Kurokawa lift

Theorem (Maaß , Andrianov, Eichler, Zagier)

Let k be even. Then for each normalized Hecke eigenform $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$, there is a Siegel eigenform $F \in S_k(\mathrm{Sp}_2(\mathbb{Z}))$ (uniquely determined up to a non-zero scalar) such that

$$L_{spin}(F, s) = \zeta(s - k + 1)\zeta(s - k + 2)L(f, s)$$

where $L(f, s)$ is the Hecke L-function of f.

- The Siegel modular form in the theorem is known as the **Saito-Kurokawa lift** of f.

Saito-Kurokawa lift

- Important arithmetic application of Saito-Kurokawa lift: **Bloch-Kato conjecture**
  - Skinner-Urban (2006): If the order of vanishing of the L-function of an ordinary modular form at the central value is odd, then the corresponding Bloch-Kato Selmer group is infinite.
  - Agarwal-Brown (2014): Evidence for Bloch-Kato conjecture
- For this application, we need generalization of Saito-Kurokawa lift
  - Skinner-Urban (2006): Saito-Kurokawa lift for a p-adic family of forms
  - Agarwal-Brown (2014): Saito-Kurokawa lift for square-free level

Hermitian Maaß lift: Level 1

Fix an imaginary quadratic field K of discriminant $-D$ i.e. $K = \mathbb{Q}(\sqrt{-D})$ and let χ be the quadratic character associated to K.

- Kojima (1980) constructed the Hermitian analogue of the Saito-Kurokawa lift for $K = \mathbb{Q}(i)$ and level $N = 1$

$$S_{k-1}(4) \longrightarrow \mathrm{M}_{k,2}(1)$$

- Krieg (1991) generalized Kojima's construction for all discriminants D and $N = 1$:

$$\mathrm{M}_{k-1}(D, \chi) \longrightarrow \mathrm{M}_{k,2}(1)$$

Hermitian Maaß lift: Level 1

- Alternative approaches (still for $N = 1$):
  - Eisenstein series by Ikeda (2008)
  - Theta lifting by Atobe (2015)
  - Converse Theorem by Matthes (2017)
- For the relation between L-function of $f \in \mathrm{M}_{k-1}(D, \chi)$ and that of the lifting $F_{f,\chi} \in \mathrm{M}_{k,2}(1)$: ignoring Euler factors dividing D,

$$L_{st}(F_{f,\chi}, s) = L(BC(f), s-2+k/2, \chi)L(BC(f), s-3+k/2, \chi)$$

Hermitian Maaß Lift

This talk: Hermitian analogue of Saito-Kurokawa lift for general level, known as the Hermitian Maaß lift in the literature

$$\mathrm{M}_{k-1}(DN,\chi) \longrightarrow \mathrm{M}^{+}_{k-1}(DN,\chi) \longrightarrow \mathrm{J}^{*}_{k,1}(N) \longrightarrow \mathrm{M}_{k,2}(N)$$

- $\mathrm{M}^{+}_{k-1}(DN,\chi) :=$ analogue of Kohnen's plus space
- $\mathrm{J}^{*}_{k,1}(N) :=$ special Hermitian Jacobi forms of level N
- $\mathrm{M}_{k,2}(N) :=$ Hermitian modular forms of level N

I will focus on my work on the middle map. The last map was constructed by Berger-Klosin (2018).

Hermitian Maaß Lift

Motivation for general level: Arithmetic applications

- Generalization of the Maaß lift to a p-adic families of forms
- Provide evidence for Bloch-Kato conjecture for more (motives coming from Galois representations associated to) modular forms (e.g. Klosin 2015)

The map $\mathrm{J}_k^*(N) \to \mathrm{M}_{k-1}^+(DN, \chi)$

Theorem (V.)

Suppose $\gcd(D, N) = 1$*. Then the map* $J_k^*(N) \to M_{k-1}^+(DN, \chi)$ *given by*

$$\phi \mapsto f := f_0|_{k-1} W_D \qquad \textit{is an injection}$$

where $W_D = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \begin{pmatrix} D & y \\ 0 & 1 \end{pmatrix}$ *and* $y = N^{-1} \bmod D$*.*

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- Here, f_u is the u^{th} theta coefficient of ϕ; $u \in [\mathfrak{D}_K] = \mathfrak{D}_K / \mathcal{O}_K$ where $\mathfrak{D}_K$ is the inverse different ideal of K
- Should be Hecke-equivariant.

The lift $\mathrm{M}^+_{k-1}(DN,\chi) \to \mathrm{J}^*_k(N)$

Theorem (V.)

Let $g \in M^+_{k-1}(DN,\chi)$. For any $u \in [\mathfrak{D}_K]$, define

$$g_u(\tau) := \chi(N)\frac{-i\sqrt{D}}{a_D(-D|u|^2)} \sum_{\substack{\ell=0 \\ \ell\equiv -D|u|^2 \bmod D}}^{\infty} a_\ell(g)\mathrm{e}\left[\frac{\ell\tau}{D}\right]$$

and

$$\phi_g := \sum_{u\in[\mathfrak{D}_K]} g_u\theta_u \quad \text{where} \quad \theta_u(\tau,z,w) := \sum_{a\in u+\mathcal{O}_K} \mathrm{e}\left[|a|^2\tau + \bar{a}z + aw\right].$$

Then $\phi_g \in J^*_{k,1}(N)$ and $\phi_g \mapsto g$ under map in previous result.

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- Unwieldy transformation formula for theta functions
  - Krieg's approach requires computation of the *inverse* of a certain matrix $M(\sigma)$ for $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ which is simple for the other generator $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $\mathrm{SL}_2(\mathbb{Z})$ but difficult for general σ

Key ideas

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- Reduce to a finite (but many) set of equations
  - Finitely many discriminants D to consider; also get rid of N

Thank you for your attention!