

# Hermitian Maaß Lift for General Level

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# Saito-Kurokawa lift

Theorem (Maaß , Andrianov, Eichler, Zagier)

*Let  $k$  be even. Then for each normalized Hecke eigenform  $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}))$ , there is a Siegel eigenform  $F \in S_k(\mathrm{Sp}_2(\mathbb{Z}))$  (uniquely determined up to a non-zero scalar) such that*

$$L_{\mathrm{spin}}(F, s) = \zeta(s - k + 1)\zeta(s - k + 2)L(f, s)$$

where  $L(f, s)$  is the Hecke  $L$ -function of  $f$ .

- ▶ The Siegel modular form in the theorem is known as the **Saito-Kurokawa lift** of  $f$ .

# Saito-Kurokawa lift

- ▶ Important arithmetic application of Saito-Kurokawa lift:  
**Bloch-Kato conjecture**
  - ▶ Skinner-Urban (2006): If the order of vanishing of the  $L$ -function of an ordinary modular form at the central value is odd, then the corresponding Bloch-Kato Selmer group is infinite.
  - ▶ Agarwal-Brown (2014): Evidence for Bloch-Kato conjecture
- ▶ For this application, we need generalization of Saito-Kurokawa lift
  - ▶ Skinner-Urban (2006): Saito-Kurokawa lift for a  $p$ -adic family of forms
  - ▶ Agarwal-Brown (2014): Saito-Kurokawa lift for square-free level

# Hermitian Maaß lift: Level 1

Fix an imaginary quadratic field  $K$  of discriminant  $-D$  i.e.  $K = \mathbb{Q}(\sqrt{-D})$  and let  $\chi$  be the quadratic character associated to  $K$ .

- ▶ Kojima (1980) constructed the Hermitian analogue of the Saito-Kurokawa lift for  $K = \mathbb{Q}(i)$  and level  $N = 1$

$$S_{k-1}(4) \longrightarrow M_{k,2}(1)$$

- ▶ Krieg (1991) generalized Kojima's construction for all discriminants  $D$  and  $N = 1$ :

$$M_{k-1}(D, \chi) \longrightarrow M_{k,2}(1)$$

# Hermitian Maaß lift: Level 1

- ▶ Alternative approaches (still for  $N = 1$ ):
  - ▶ Eisenstein series by Ikeda (2008)
  - ▶ Theta lifting by Atobe (2015)
  - ▶ Converse Theorem by Matthes (2017)
- ▶ For the relation between  $L$ -function of  $f \in M_{k-1}(D, \chi)$  and that of the lifting  $F_{f, \chi} \in M_{k, 2}(1)$ : ignoring Euler factors dividing  $D$ ,

$$L_{st}(F_{f, \chi}, s) = L(BC(f), s - 2 + k/2, \chi)L(BC(f), s - 3 + k/2, \chi)$$

# Hermitian Maaß Lift

*This talk:* Hermitian analogue of Saito-Kurokawa lift for general level, known as the Hermitian Maaß lift in the literature

$$M_{k-1}(DN, \chi) \longrightarrow M_{k-1}^+(DN, \chi) \longrightarrow J_{k,1}^*(N) \longrightarrow M_{k,2}(N)$$

- ▶  $M_{k-1}^+(DN, \chi) :=$  analogue of Kohnen's plus space
- ▶  $J_{k,1}^*(N) :=$  special Hermitian Jacobi forms of level  $N$
- ▶  $M_{k,2}(N) :=$  Hermitian modular forms of level  $N$

I will focus on my work on the middle map. The last map was constructed by Berger-Klosin (2018).

# Hermitian Maaß Lift

*Motivation for general level:* Arithmetic applications

- ▶ Generalization of the Maaß lift to a  $p$ -adic families of forms
- ▶ Provide evidence for Bloch-Kato conjecture for more (motives coming from Galois representations associated to) modular forms (e.g. Klosin 2015)

# The map $J_k^*(N) \rightarrow M_{k-1}^+(DN, \chi)$

## Theorem (V.)

Suppose  $\gcd(D, N) = 1$ . Then the map  $J_k^*(N) \rightarrow M_{k-1}^+(DN, \chi)$  given by

$$\phi \mapsto f := f_0|_{k-1} W_D \quad \text{is an injection}$$

where  $W_D = \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix} \begin{pmatrix} D & y \\ 0 & 1 \end{pmatrix}$  and  $y = N^{-1} \pmod{D}$ .



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- ▶ Here,  $f_u$  is the  $u^{\text{th}}$  theta coefficient of  $\phi$ ;  $u \in [\mathfrak{D}_K] = \mathfrak{D}_K/\mathcal{O}_K$  where  $\mathfrak{D}_K$  is the inverse different ideal of  $K$
- ▶ Should be Hecke-equivariant.

# The lift $M_{k-1}^+(DN, \chi) \rightarrow J_k^*(N)$

## Theorem (V.)

Let  $g \in M_{k-1}^+(DN, \chi)$ . For any  $u \in [\mathfrak{D}_K]$ , define

$$g_u(\tau) := \chi(N) \frac{-i\sqrt{D}}{a_D(-D|u|^2)} \sum_{\substack{\ell=0 \\ \ell \equiv -D|u|^2 \pmod{D}}}^{\infty} a_\ell(g) e\left[\frac{\ell\tau}{D}\right]$$

and

$$\phi_g := \sum_{u \in [\mathfrak{D}_K]} g_u \theta_u \quad \text{where} \quad \theta_u(\tau, z, w) := \sum_{a \in u + \mathcal{O}_K} e\left[|a|^2\tau + \bar{a}z + aw\right].$$

Then  $\phi_g \in J_{k,1}^*(N)$  and  $\phi_g \mapsto g$  under map in previous result.

## Main technical difficulties

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- ▶ Unwieldy transformation formula for theta functions
  - ▶ Krieg's approach requires computation of the *inverse* of a certain matrix  $M(\sigma)$  for  $\sigma \in SL_2(\mathbb{Z})$  which is simple for the other generator  $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  of  $SL_2(\mathbb{Z})$  but difficult for general  $\sigma$

## Key ideas

- ▶ Reduce the verification of the functional equation  $\phi_g |_{k,1} \sigma = \phi_g$  to verification of a collection of “arithmetic equations” (for each  $\sigma \in \Gamma_0(N)$ )



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- ▶ Reduce to a finite (but many) set of equations
  - ▶ Finitely many discriminants  $D$  to consider; also get rid of  $N$

Thank you for your attention!