Riemann Surfaces

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In this note, I am going to recall a bit of the theory of Riemann surfaces. Familiarity with sheaf, algebraic/differential geometry and complex analysis is assumed.

1. The objects

The theory of Riemann surface is basically 1-dimensional $\mathbb{C}$-version of differential geometry so if you are familiar with differential geometry, you should be able to formulate part of the theory.

Definition 1.1. A *Riemann surface* is a one dimensional smooth complex manifold.

More concretely, a Riemann surface

$$\mathcal{R} = (R; \{(U_\alpha, z_\alpha)\}_{\alpha \in I})$$

consists of

- a *connected* complex manifold R;
- an open covering $\{U_\alpha\}_{\alpha \in I}$ of R where I is some indexing set; and
- homeomorphism $z_\alpha : U_\alpha \to V_\alpha$ where V_α is open subset of $\mathbb{C}$

such that for every $\alpha, \beta \in I$ such that $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, the *transition maps*

$$z_\beta \circ z_\alpha^{-1} : z_\alpha(U_{\alpha\beta}) \to z_\beta(U_{\alpha\beta})$$

and $z_\beta \circ z_\alpha^{-1}$ are (the usual) holomorphic function (from open subset of $\mathbb{C} \to \mathbb{C}$).

Such a collection $(U_\alpha, z_\alpha)_\alpha$ is called a *coordinate chart* on $\mathcal{R}$.

Remark 1.2. Think of z_α as coordinate function or as the "complex variable" on U_α. An alternative way to define a Riemann surface is a collection $\{(V_\alpha, V_{\alpha\beta}, t_{\alpha\beta})\}$ where $V_{\alpha\beta} \subset$

$V_\alpha \subseteq \mathbb{C}$ are open in $\mathbb{C}$ and $t_{\alpha\beta} : V_{\alpha\beta} \to V_{\beta\alpha}$ are holomorphic transition maps. Then we can build a surface $R = \bigsqcup V_\alpha / \sim$ where V_α and V_β are glued along their subset $V_{\alpha\beta} \leftrightarrow V_{\beta\alpha}$.

Example 1.3. Any connected open subset U of $\mathbb{C}$, in particular the upper half plane

$$\mathfrak{H} := \{z \in \mathbb{C} | \Im(z) > 0\}$$

or the whole $\mathbb{C}$, is a Riemann surface with chart $\{U, \mathrm{Id}\}$.

Example 1.4. A basic non-trivial example of Riemann surfaces is the Riemann sphere $\mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ whose chart consists of

$$\begin{aligned} U_0 &= \mathbb{C} \\ z_0 &= \mathrm{Id} : z \mapsto z \end{aligned}$$

and

$$U_1 = \mathbb{P}^1_{\mathbb{C}} - \{0\} = \mathbb{C}^\times \cup \{\infty\}$$

$$z_1 : z \mapsto \begin{cases} \frac{1}{z} & \text{if } z \in C^\times, \\ 0 & \text{if } z = \infty. \end{cases}$$

From now on, when we refer to $\mathbb{C}$ or $\mathfrak{H}$ or $\mathbb{P}^1_{\mathbb{C}}$, the Riemann surface structure is implicitly understood to be from the above examples.

Definition 1.5. Two charts (U_α, z_α) and (U'_i, z'_i) for a surface R are *equivalent* if their union also forms a chart for R.

Two Riemann surface with equivalent chart are considered to be the same object.

Exercise 1. Show that any Riemann surface has a maximal chart and that chart is unique.

Remark 1.6. A chart can have repetition. All it takes for (U, z) and (U, w) to be in the same chart is that $w \circ z^{-1}$ is holomorphic!

The maximal chart has a lot of such repetition. Many results can be proved by local analysis: taking an arbitrary local chart (U, z) in the maximal chart and consider the classical complex analytic object locally on $z(U)$.

2. The morphisms

Definition 2.1. Let $U \subset \mathcal{R}$ be open. A function $f : U \to \mathbb{C}$ is called *holomorphic*[1] if and only if $f \circ z_\alpha^{-1} : z_\alpha(U \cap U_\alpha) \to \mathbb{C}$ is holomorphic for all α.

We denote by $\mathcal{O}_\mathcal{R}(U)$ the ring of all holomorphic functions on U.

[1] I should define only holomorphic function $\mathcal{R} \to \mathbb{C}$; since any open subset $U \subset \mathcal{R}$ inherits the charts from $\mathcal{R}$ and thus is automatically a Riemann surface.

Note that $\mathcal{O}_{\mathcal{R}}(U) \subset \mathcal{C}_{\mathcal{R}}(U)$ where $\mathcal{C}_{\mathcal{R}}(U)$ denotes the ring of all continuous[2] functions $U \to \mathbb{C}$.

Remark 2.2. In term of "data", a holomorphic function f is a collection of (usual complex) holomorphic functions $f_\alpha(z_\alpha) : V_\alpha \to \mathbb{C}$ satisfying the compatibility condition (a.k.a. sheaf condition) i.e. such that $f_\alpha \circ z_\alpha^{-1} = f_\beta \circ z_\beta^{-1}$ when restricting to $U_{\alpha\beta}$.

The analogy with algebraic geometry is to give a global section $\Gamma(X, \mathfrak{F})$ on a sheaf $\mathfrak{F}$ is the same as giving a collection of compatible local sections $s_\alpha \in \Gamma(U_\alpha, \mathfrak{F})$ for an open cover $\{U_\alpha\}$ of X.

As illustration, to give a holomorphic function on $\mathbb{P}^1_{\mathbb{C}}$ is the same as giving two holomorphic functions $f_0, f_1 : \mathbb{C} \to \mathbb{C}$ such that $f_0(z) = f_1(\frac{1}{z})$ for all $z \neq 0$. (From here, it

[2] $\mathcal{C}_{\mathcal{R}}(U)$ does not depend on R; we just add it for clarity.

should be obvious that the only holomorphic functions on $\mathbb{P}^1_{\mathbb{C}}$ are constants.)

For future reference, to save time, we introduce many notions (e.g. differential forms, divisors, etc.) in term of a chart; which is essentially defining a section of a sheaf (e.g. sheaf of differential forms, sheaf of divisors, etc.) explicitly by gluing local sections. To put it another way, we define many sheaves $\mathfrak{F} = \mathcal{O}, \mathcal{K}, ...$ by gluing smaller sheaves $\mathfrak{F}_\alpha$ on a covering U_α. Now, a problem that arised is one needs to have a way of relating sections describing via different covering; for one open set can have many covering. For instance, if $U = \bigcup U_\alpha = \bigcup V_i$ then (U_α, s_α) could give the same section on U as (V_i, t_i): they are the same exactly when they are compatible i.e. $\{U_\alpha\} \cup \{V_i\}$ also works. So we define two objects (defined with charts) to be equivalent if their combination is also an object. For instance, if $\omega = (U_\alpha, z_\alpha, f_\alpha)$ and $\omega' = (V_i, w_i, g_i)$

are differentials then ω is equivalent to ω' if their union is also a differential.

We can relativize the concept of holomorphic functions, which we typically refer to as holomorphic maps:

Definition 2.3. Let $\mathcal{R}, \mathcal{R}'$ be Riemann surfaces. A continuous map $F : \mathcal{R} \to \mathcal{R}'$ is *holomorphic* if and only if

$$F_U^*(\mathcal{O}_{\mathcal{R}'}(U)) \subseteq \mathcal{O}_{\mathcal{R}}(F^{-1}(U))$$

for every open $U \subset \mathcal{R}'$.

Here, the natural *pull-back map*

$$F_U^* : \mathcal{C}_{\mathcal{R}'}(U) \to \mathcal{C}_{\mathcal{R}}(F^{-1}(U))$$

is given by $\varphi \mapsto \varphi \circ F$.

With the last definition, we have a category of Riemann surfaces whose morphisms are holomorphic maps. As a remark, a holomorphic function $\mathcal{R} \to \mathbb{C}$ in Definition 2.1 is a holomorphic map $\mathcal{R} \to \mathbb{C}$ in Definition 2.3 where $\mathbb{C}$ is viewed as a Riemann surface!

Exercise 2. Show that holomorphic function doesn't depend on the choice of an equivalent chart[3]. In other words, if (U_α) and (V_i) are equivalent chart on $\mathcal{R}$ then f is holomorphic with respect to (U_α) if and only if it is holomorphic with respect to (V_i).

Exercise 3. Verify that $(\mathcal{R}, \mathcal{O}_{\mathcal{R}})$ is a ringed space. In particular, $\mathcal{O}_{\mathcal{R}}$ is a sheaf of rings on $\mathcal{R}$. Verify that similar constructions $\mathcal{C}_{\mathcal{R}}$, $\mathcal{C}^\infty$ also defines sheaves: this should be obvious since our sections are functions and restrictions are restriction of functions.

Also, suppose that (U_α, z_α) is a chart. Show that the sheaf $\mathcal{O}_{\mathcal{R}}$ could be obtained by gluing the sheaves $\mathcal{O}_\alpha$ on U_α where for any $U \subseteq U_\alpha$

[3]Because of this, an alternative way of defining a Riemann surface is as a complex manifold + a sheaf of holomorphic function.

open, $\mathcal{O}_\alpha(U)$ is the ring of (usual) holomorphic functions on $z_\alpha(U) \subseteq \mathbb{C}$.

Exercise 4. What are the stalks $\mathcal{O}_P$ as rings? (*Answer:* Observe that a stalk at P is a local object i.e. only depend on a chart at P. Let (U, z) any chart with $P \in U$; then any other stalk $[V, f] \in \mathcal{O}_P$ can be identified with $[V \cap U, f|_{V \cap U}] \in \bigcup_{P \in W \subset U} \mathcal{O}_P(W)$. Then a holomorphic function at P can be identified with a power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ where $z_0 = z(P)$ having positive radius of convergent. Thus, $\mathcal{O}_P$ can be identified with the subring of the power series ring $\mathbb{C}[[z]]$ consisting of those with *positive* radius of convergence.)

With this, we can easily define the notion of *meromorphic function* on $\mathcal{R}$ to be holomorphic function $\mathcal{R} \to \mathbb{P}^1_{\mathbb{C}}$ that is not constant ∞ and likewise *meromorphic maps* $\mathcal{R} \to \mathcal{R}'$ between two Riemann surfaces. Similar to the sheaf

$\mathcal{O}_{\mathcal{R}}$, we have a sheaf $\mathcal{K}_{\mathcal{R}}$ where $\mathcal{K}_{\mathcal{R}}(U)$ is the field[4] of meromorphic functions on U.

Exercise 5. Define meromorphic function using chart. Be careful here that meromorphic function (when viewed as $\mathbb{C}$-valued function) are NOT defined on the whole surface! Prove that the above definition is equivalent to it.

Exercise 6. What are the stalks of $\mathcal{K}$? (*Answer*: Field of fractions of $\mathcal{O}_P$.)

Show that there is an exact sequence

$$0 \to \mathcal{O}_P^\times \to \mathcal{K}_P^\times \to \mathbb{Z} \to 0$$

where the second map returns the order of vanishing of f at P.

Exercise 7. Verify that $\mathcal{K}(U)$ is a field if U is connected.

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[4]A field only when U is connected! Otherwise, it is product of fields.

3. Simply connected Riemann surfaces

It is well-known from topology that every (reasonable) space has the universal covering space. For every Riemann surface $\mathcal{R}$, its universal covering space $\widetilde{\mathcal{R}}$ has a natural structure of a Riemann surface. Thus, we want to classify all simply connected Riemann surface.

Theorem 3.1 (Uniformization)**.** *There are only three simply connected Riemann surface (upto isomorphism): the complex plane* $\mathbb{C}$*, the Riemann sphere* $\mathbb{P}^1_{\mathbb{C}}$ *and the upper half plane* $\mathfrak{H}$*.*

The Riemann surface structures are as in previous examples. Note in particular that the theorem implies that there is no other Riemann surface structures on these spaces.

It follows then that any Riemann surface is the quotient of its universal covering space (one of $\mathbb{P}^1_{\mathbb{C}}, \mathbb{C}, \mathfrak{H}$) by a group, namely its *fundamental group*[5]. (See [6], Chapter I, Section 1.2 for some basic facts about topological group actions.)

Exercise 8. This exercise is to review simple results from general topology.

(i) Prove that connected + locally path connected $\Rightarrow$ path connected.

(ii) On a path connected space X, recall that the fundamental groups

$$\pi_1(X, x_0) \cong \pi_1(X, x_1)$$

for any $x_0, x_1 \in X$. (Proposition 1.5 of [4].)

(iii) Find the natural action of $\pi_1(X, x_0)$ on the universal cover $\widetilde{X}$. (Use results of Section 1.3 in [4]. In fact, recall that the points of $\widetilde{X}$ are paths γ on X starting

[5]Think of the circle as quotient $\mathbb{R}/\mathbb{Z}$ where $\mathbb{R}$ is its universal cover and $\mathbb{Z}$ is its fundamental group!

from x_0 up to homotopy so the natural action is simply path concatenation!)

(iv) Deduce that $X \cong \widetilde{X}/\pi_1(X, x_0)$.

Recall from basic complex analysis (see [1]) that $\mathfrak{H}$ is bi-holomorphic to the unit disk and that its automorphism group (in the category of Riemann surfaces) is precisely the group of real Möbius (i.e. linear fractional) transformation $\mathsf{PSL}_2(\mathbb{R}) = \mathsf{SL}_2(\mathbb{R})/\pm I$ (Theorem 1.1.3 of [6], essentially Schwarz lemma); in other words and more generally, the group $\mathsf{GL}_2(\mathbb{R})^+$ acts on $\mathfrak{H}$ naturally by linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}. \tag{3.1}$$

This action is transitive as

$$\frac{1}{\sqrt{y}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \cdot i = x + iy.$$

(Note: We have $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \mathsf{GL}_2(\mathbb{R})^+$ because $x + iy \in \mathfrak{H}$.)

This allows us to identify

$$\mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2(\mathbb{R}) \cong \mathfrak{H}$$

as smooth manifold since $\mathsf{SO}_2(\mathbb{R}) = \mathsf{SL}_2(\mathbb{R})_i$ is the isotropy group of i (via Theorem 1.2.1 of [6]).

From $\mathsf{SL}_2(\mathbb{R})$ being Lie group, $\mathfrak{H}$ should have a Haar measure i.e. $\mathrm{Aut}(\mathfrak{H})$-invariant volume form. It also have an $\mathrm{Aut}(\mathfrak{H})$-invariant metric. We will talk about geometry on $\mathfrak{H}$ in a different note.

Exercise 9. What are the automorphism groups of the other simply connected Riemann surfaces i.e. $\mathbb{C}$ and $\mathbb{P}^1_{\mathbb{C}}$? (*Answer*: $\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b | a, b \in \mathbb{C}\}$ and $\mathrm{Aut}(\mathbb{P}^1_{\mathbb{C}}) = \mathsf{PSL}_2(\mathbb{C})$.)

Exercise 10. Verify that (3.1) defines a group action; in particular, it is well-defined $\gamma z \in \mathfrak{H}$ if $z \in \mathfrak{H}$.

It follows from the determination of automorphism groups that Riemann surfaces whose universal cover is $\mathfrak{H}$ is isommorphic to $\mathfrak{H}/\Gamma$ for some subgroup $\Gamma \subset \mathsf{PSL}_2(\mathbb{R})$.

4. Compact Riemann Surface

Compact Riemann surfaces have a good theory. It was attributed to Riemann that

Theorem 4.1 (Riemann's Existence Theorem)**.** *Any compact Riemann surface is a projective variety (an algebraic curve).*

According to Hartshorne, Riemann's first step of the proof is to show that there exists a non-constant meromorphic function on a compact Riemann surface (hence the name "Existence Theorem"). Afterwards, we have from algebraic geometry that a smooth projective curve is completely determined by its function field.

Let $K(\mathcal{R})$ denote the *field of meromorphic function* on $\mathcal{R}$. Thanks to Theorem 4.1, this is a field extension of $\mathbb{C}$ of transcendance degree 1. Hence, if $\varphi \in K(\mathcal{R})$ is non-constant i.e. $\varphi \notin \mathbb{C}$ then φ is transcendental and $K(\mathcal{R})/\mathbb{C}(\varphi)$ is a finite extension whose degree is exactly the number of zeros (with multiplicity) of φ; or equivalently, the number of poles (c.f. equation (1.8.6) in [6]).

Remark 4.2. It should not be hard to see the reason for the above statement: The field injection $\mathbb{C}(\varphi) \to K(\mathcal{R})$ gives rise to rational map $\mathcal{R} \to \mathbb{P}^1_{\mathbb{C}}$. Then the number of poles can be viewed as the number of pre-images of $\infty \in \mathbb{P}^1_{\mathbb{C}}$ and the number of zeros is just the number of pre-images of $0 \in \mathbb{P}^1_{\mathbb{C}}$. These number are equals; for we expect the number of pre-images of any other point on $\mathbb{P}^1_{\mathbb{C}}$ is the same. This common number is typically defined as the *degree of the covering* $\mathcal{R} \to \mathbb{P}^1_{\mathbb{C}}$ in differential geometry or topology.

Theorem 4.3 (Hurwitz's Formula)**.** *Let $\mathcal{R}$ and $\mathcal{R}'$ be compact Riemann surfaces of [topological] genus g and g' respectively and let $F : \mathcal{R}' \to \mathcal{R}$ is a degree n covering. Then*

$$2g' - 2 = n(2g - 2) + \sum_{P \in \mathcal{R}'} (e_{P,F} - 1).$$

Here, the degree of a covering can be determined algebraically as follow: as with before, the pull-back map $F^* : K(\mathcal{R}) \to K(\mathcal{R}')$ define a field homomorphism so $K(\mathcal{R}')$ is a field extension of $K(\mathcal{R})$. The degree of the covering is exactly the degree of this extension. (This should be the same as the analytic definition of degree; via integration using volume form for instance.)

The numbers $e_{P,F}$'s are *ramification index* at the point $P \in \mathcal{R}'$ i.e. the multiplicity of the point P lying above $F(P)$; and is defined as the order of the (usual) meromorphic function $z \circ F \circ w^{-1} : w(U) \to V$ at $w(P) \in \mathbb{C}$ if we choose local chart (V, z) in $\mathcal{R}'$ at $F(P)$ and local chart (U, w) in $\mathcal{R}'$ at P such that $U \subset F^{-1}(V)$.

5. Differential Forms

One can talk about 1-form (or n-forms) on Riemannian manifold as in differential geometry. One can give a sheaf-like definition but this requires redoing differential geometry (tangent space, cotangent space, exterior product, etc.); see [2], Chapter I, Section 9 for the details. To save time here, let me do it concretely in term of charts which is again just describing a global section from compatible local sections.

Definition 5.1. A holomorphic (meromorphic) *degree-k differential* (a degree-k 1-form) on an open subset $U \subset \mathcal{R}$,

$$\omega = (U_\alpha, z_\alpha, f_\alpha)$$

consists of

- a coordinate chart (U_α, z_α) covering U; and
- a collection $(f_\alpha : U_\alpha \to \mathbb{P}^1_{\mathbb{C}})$ of holomorphic (meromorphic, resp.) functions such that (the sheaf condition)

$$f_\alpha(P)\left(\frac{dz_\alpha}{dz_\beta}\right)^k(P) = f_\beta(P)$$

  holds for every $P \in U_{\alpha\beta}$.

If you rearrange the equation "formally", it becomes more mnemonic

$$f_\alpha(dz_\alpha)^k = f_\beta(dz_\beta)^k$$

but bear in mind that we do not have any formal treatment of $f_\alpha(dz_\alpha)^k$! So in analogy with the concept of "holomorphic functions", a differential is just a bunch of $f_\alpha(dz_\alpha)^k$ satisfying compatibility (i.e. sheaf) conditions.

Remark 5.2. The equation in Definition 5.1 needs some clarification on the term $\left(\frac{dz_\alpha}{dz_\beta}\right)^k(P)$. Here, let

$$V_\beta := z_\beta(U_{\beta\alpha}) \subset \mathbb{C}$$
$$V_\alpha := z_\alpha(U_{\alpha\beta}) \subset \mathbb{C}$$

Note that $P = z_\beta(w_0)$ for some $w_0 \in V_\beta$. Then we have transition map

$$t := z_\alpha \circ z_\beta^{-1} : V_\beta \to V_\alpha$$

and then

$$\left(\frac{dz_\alpha}{dz_\beta}\right)^k (P) := (t'(w_0))^k \in \mathbb{C}.$$

Remark 5.3. A degree-1 differential is an 1-form in differential geometry

$$(U_\alpha, z_\alpha = x_\alpha + iy_\alpha, f_\alpha \, dx_\alpha + g_\alpha \, dy_\alpha)$$

satisfying some equations indicating that it is "meromorphic". (See [2] for details. As an analogy, observe that a holomorphic function are basically a pair of $\mathcal{C}^\infty$ function satisfying Cauchy-Riemann equation.) The number $(t'(w_0))$ in previous remark is nothing other than the transition requirement enforced by the Jacobian.

Denote $D^k(\mathcal{R})$ the $\mathbb{C}$-vector space of degree k differentials. Observe that

(i) $\bigoplus_{k\in\mathbb{Z}} D^k(\mathcal{R})$ is a graded ring where sum and product are chart-wise;

(ii) if $\omega = (U_\alpha, z_\alpha, f_\alpha) \in D^k(\mathcal{R})$ is non-zero then

$$\omega^{-1} = (U_\alpha, z_\alpha, f_\alpha^{-1}) \in D^{-k}(\mathcal{R});$$

(iii) $D^0(\mathcal{R}) = K(\mathcal{R})$ and there is differentiation $D^0(\mathcal{R}) \to D^1(\mathcal{R})$ given by

$$\varphi = (\varphi_\alpha) \mapsto d\varphi := \left(\frac{d\varphi_\alpha}{dz_\alpha}\right);$$

(iv) $D^k(\mathcal{R})$ are 1 dimensional $K(\mathcal{R})$ vector spaces (assuming that $\mathcal{R}$ is compact): Since $D^0(\mathcal{R}) = K(\mathcal{R})$ is field extension of $\mathbb{C}$ of transcendence degree 1, pick any non-constant $\varphi \in K(R)$ and we find that $0 \neq (d\varphi)^k \in D^k(\mathcal{R})$. Then for any $\omega \in D^k(\mathcal{R})$, we find that $\omega(d\varphi)^{-k} \in D^0(\mathcal{R})$. Hence,

$$\omega \in K(\mathcal{R})(d\varphi)^k$$

or in other words,

$$D^k(\mathcal{R}) = K(\mathcal{R})(d\varphi)^k$$

is an one dimensional $K(\mathcal{R})$ vector space.

6. Divisors

Definition 6.1. A *divisor* is a global section of the sheaf $\mathcal{K}^\times/\mathcal{O}^\times$.

Remark 6.2. Again, we could describe a divisor as $D = (U_\alpha, z_\alpha, f_\alpha)$ with $f_\alpha \in \mathcal{K}^\times(U_\alpha)$

and if $U_{\alpha\beta} \neq \emptyset$ then there exists a holomorphic function $g \in \mathcal{O}^\times(U_{\alpha\beta})$ such that $\frac{1}{g}$ is also holomorphic and $f_\alpha = gf_\beta$ on $U_{\alpha\beta}$. In other words, f_α/f_β has neither pole nor zero in $U_{\alpha\beta}$.

The reason $\mathcal{K}^\times/\mathcal{O}^\times$ comes up is because it sits in the exact sequence

$$0 \to \mathcal{O}^\times \to \mathcal{K}^\times \to \mathcal{K}^\times/\mathcal{O}^\times \to 0.$$

And we are interested in sheaf cohomology $H^1(\mathcal{R}, \mathcal{O}^\times)$, which is important in the theory of Riemann surface.

6.1. **Interpretation.** I shall give some interpretation for the notion of divisors.

On a *compact* Riemann surface, there is an alternative (and more familiar) method to give divisors: as finite formal sum $\sum n_P \cdot P$.

The idea is that the stalks $(\mathcal{K}^\times/\mathcal{O}^\times)_P = \mathcal{K}_P^\times/\mathcal{O}_P^\times$ could be easily determined to be $\mathbb{Z}$ (via the previous exercises) and so a section can be view as a mapping $\mathcal{R} \to \mathbb{Z}$. Now due to compactness, we must have finitely many non-zero since a meromorphic function on a compact $\mathcal{R}$ can only have finitely many poles/zeros.

Alternative, by the interpretation in the remark and the fact that any cover reduces to a finite cover, one sees that there is a finite set $(U_1, z_1, f_1), ..., (U_n, z_n, f_n)$ that determines the divisor. Since meromorphic has same poles as zeros (with multiplicity), one has $\sum n_P = 0$.

6.2. **Divisor associated to a differential.**

7. Final Exercise

Exercise 11. Read Ahlfors' [1] and Forster's [2] and many other books like [3] to learn Riemann surfaces properly. Miyake [6] refers to Lang [5] for a treatment of Riemann surface.

References

[1] Ahlfors, L. V. *Complex analysis*, third ed. McGraw-Hill Book Co., New York, 1978. An introduction to the theory of analytic functions of one complex variable, International Series in Pure and Applied Mathematics.

[2] Forster, O. *Lectures on Riemann surfaces*, vol. 81 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. Translated from the 1977 German original by Bruce Gilligan, Reprint of the 1981 English translation.

[3] Gunning, R. C. *Lectures on Riemann surfaces*. Princeton Mathematical Notes. Princeton University Press, Princeton, N.J., 1966.

[4] Hatcher, A. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[5] Lang, S. *Introduction to algebraic and abelian functions*, second ed., vol. 89 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[6] Miyake, T. *Modular forms*, english ed. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2006. Translated from the 1976 Japanese original by Yoshitaka Maeda.