Modular Forms and Complex Analysis II

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In this note, we are going to introduce the modular curves and give an interpretation of modular forms as differential forms on modular curves. With this geometric interpretation, we can find the dimension of the space of modular forms using Riemann-Roch theorem.

1. Modular Curves

Fix Γ and denote the Γ-extended upper half plane

$$\mathfrak{H}_\Gamma := \mathfrak{H} \cup \{\text{cusps of } \Gamma\}.$$

(Many authors denote $\mathfrak{H}^*$ for $\mathfrak{H}_\Gamma$ but I don't like this since it doesn't highlight the dependency of $\mathfrak{H}^*$ on Γ.)

Definition 1.1. The *modular curve* of level Γ is the quotient

$$X_\Gamma := \mathfrak{H}_\Gamma/\Gamma.$$

We also denote

$$Y_\Gamma := \mathfrak{H}/\Gamma.$$

We also denote $X_\Box(N) := X_{\Gamma_\Box(N)}$ where $\Box$ can be 0, 1 or nothing.

At this point, X_Γ is just a set. We are going to introduce a topology as well as a complex chart to make it into a Riemann surface.

1.1. Topology on $\mathfrak{H}_\Gamma$.

We take a basis for the topology on $\mathfrak{H}_\Gamma$ consisting of

- Usual fundamental system of neighborhood for each point $z \in \mathfrak{H}$; and

- At each cusp x, pick $\sigma \in \mathsf{SL}_2(\mathbb{R})$ such that $\sigma x = \infty$ and take $\{\sigma^{-1} U_l^* \,|\, l > 0\} \cap \mathfrak{H}_\Gamma$ as basic open sets where

$$U_l^* := \{z \in \mathfrak{H} \,|\, \Im(z) > l\} \cup \{\infty\}.$$

(Note that the $\{U_l^*\}$ forms a system of neighborhood of ∞ should it be a cusp of Γ.)

Exercise 1. Show that $\mathfrak{H}_\Gamma$ is Hausdorff with this topology.

With the topology on $\mathfrak{H}_\Gamma$, we have the natural quotient topology on X_Γ.

Exercise 2. (i) Show that if a group G acts properly discontinuously on a Hausdorff space X then X/G is Hausdorff. (Recall that G acts *properly discontinuously* on X if for any two points $x, y \in X$ including $x = y$, there exists neighborhood $U \ni x, V \ni y$ such that

$$\{g \in G \,|\, gU \cap V \neq \emptyset\}$$

is finite. In words, V intersects only finitely many G-translates of U.)

(ii) Show that X_Γ is Hausdorff.

Exercise 3. Is it possible to make $\mathfrak{H}_\Gamma$ into a Riemann surface?

1.2. **Modular curve as Riemann surface.** Now we introduce a Riemann surface structure on X_Γ.

Let $\pi = \pi_\Gamma : \mathfrak{H}_\Gamma \to X_\Gamma$ be the projection map and recall that the topology on X_Γ is the finest one such that π is continuous.

We call a point $P \in X_\Gamma$ an elliptic point, a cusp or an ordinary point if $P = \pi(\tau)$ and $\tau \in \mathfrak{H}_\Gamma$ is correspondingly an elliptic point, a cusp or otherwise.

A chart on X_Γ is given by $(U_P, z_P)_{P \in X_\Gamma}$ where

(i) We pick a representative $\tau_P \in \mathfrak{H}_\Gamma$ (a lift of P) for each $P \in X_\Gamma$. We will drop the subscript if there is no confusion.

(ii) If P is an ordinary point, let $V_\tau \subset \mathfrak{H}$ be neighborhood of τ such that

$$\gamma V_\tau \cap V_\tau \neq \emptyset \Rightarrow \gamma \in \{\pm I\} = \pm\Gamma_\tau.$$

Such U exists due to Γ acting properly discontinuously on $\mathfrak{H}_\Gamma$. (Note that $\pm I$ gives the same identity action on $\mathfrak{H}_\Gamma$.)

We take

$$U_P = \pi(V_\tau)$$

and

$$z_P = \pi^{-1} : U_P \to V_\tau$$

as the coordinate chart at P.

(iii) For every elliptic point P, let $\rho \in \mathsf{SL}_2(\mathbb{C})$ be the Möbius transformation of $\mathfrak{H}$ to the unit disk $\mathfrak{D}_1$ such that $\tau \mapsto 0$.

Recall that Γ_τ is finite cyclic group. So is $\rho\Gamma_\tau\rho^{-1} \subset \mathrm{Aut}(D)$. Consequently, elements of $\rho\Gamma_\tau\rho^{-1}$ are all rotations of angle $\frac{2\pi n}{e}$ where e is the order of the group.

Again, using the properly discontinuously action, we pick sufficiently small radius $r < 1$ such that the pre-image (under ρ) of the disk of radius r, namely

$$\mathfrak{D}_r := \{z \in \mathbb{C} \mid |z| < r\}$$

gives rise to neighborhood $V_\tau = \rho^{-1}(\mathfrak{D}_r) \subset \mathfrak{H}$ such that

$$\gamma V_\tau \cap V_\tau \Rightarrow \gamma \in \Gamma_\tau.$$

Then we take $U_P = \pi(V_\tau)$ and

$$z_P : U_P \to \mathfrak{D}_{r^e}$$

as the composition of homeomorphisms

$$U_P \to \Gamma_\tau \backslash V_\tau \to \rho \Gamma_\tau \rho^{-1} \backslash \mathfrak{D}_r \to \mathfrak{D}_{r^e}$$

where the last map is raising to power e i.e. $w \mapsto w^e$.

(iv) If P is a cusp, let $\sigma \in \mathsf{SL}_2(\mathbb{R})$ be such that $\sigma\tau = \infty$.

As before, we choose l sufficiently large so that the neighborhood $V_\tau = \sigma^{-1} U_l^*$

satisfies

$$\gamma V_\tau \cap V_\tau \Rightarrow \gamma \in \Gamma_\tau.$$

(Recall that Γ_τ is essentially infinite cyclic.)

Then we take the chart $U_P = \pi(V_\tau)$ and

$$z_P : U_P \to \mathfrak{D}_r$$

as composition

$$U_P \to \Gamma_\tau \backslash V_\tau \to \sigma\Gamma_\tau\sigma^{-1} \backslash U_l^* \to \mathfrak{D}_r$$

where the last map is $z \mapsto e^{2\pi i z/h}$ if $z \in U_l^* - \{\infty\}$ and 0 for ∞; just to get us from $U_l^* = \{z \in \mathbb{C} | \Im(z) > l\} \cup \{\infty\}$ which is not a subset of $\mathbb{C}$ to some disk $\mathfrak{D}_r \subset \mathbb{C}$.

So to sum up, we choose a neighborhood V_τ so that

$$\gamma V_\tau \cap V_\tau \neq \emptyset \Rightarrow \gamma \in \Gamma_\tau$$

and then set $U_P := \pi(V_\tau)$ and the map z_P appropriately.

We are going to be interested in Fuchsian groups Γ so that X_Γ is compact (and hence, admits good Riemann surface theory). Such Γ are called *Fuchsian group of the first kind*. A worthy to mention result is that

Theorem 1.2 (Siegel, Theorem 1.9.1 in [1]). *A Fuchsian group Γ is of the first kind if and only if the (hyperbolic) volume of X_Γ is finite.*

Exercise 4. What are all the matrices in $\mathsf{SL}_2(\mathbb{C})$ that maps $\mathfrak{H}$ to the unit disk? Determine explicitly the one that maps τ to 0 for any $\tau \in \mathfrak{H}$. (*Answer*: The matrix

$$\underbrace{\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}}_{\mathfrak{H}\to\mathfrak{D}_1 \quad i\mapsto 0} \underbrace{\begin{pmatrix} \frac{1}{\sqrt{\Im(\tau)}} & 0 \\ 0 & \sqrt{\Im(\tau)} \end{pmatrix} \begin{pmatrix} 1 & -\Re(\tau) \\ 0 & 1 \end{pmatrix}}_{\mathfrak{H}\to\mathfrak{H} \quad \tau\mapsto i}$$

should work.)

Exercise 5. Most books don't do this but show that the chart just defined is a valid chart and define a Riemann surface structure on X_Γ. (*Answer*: We need to show that if $P, Q \in X_\Gamma$ such that $U_{PQ} = U_P \cap U_Q \neq \emptyset$ then the transition map

$$z_P(U_{PQ}) \to z_Q(U_{PQ})$$

is holomorphic. This problem is local so we reduce to the case where $Q \in U_P$. If P is ordinary, Q must be ordinary as well since $V_{\tau_P} \subset \mathfrak{H}$ consists of ordinary points; and let τ'_Q be the pre-image of Q in V_{τ_P} then we have $\tau'_Q = \gamma(\tau_Q)$ for some $\gamma \in \Gamma$ since both represents Q and the transition map is exactly given by γ, which is holomorphic. If P is an elliptic point, then the transition is a composition of z^m and $\mathsf{SL}_2(\mathbb{C})$. If P is a cusp, then it is composition of exponential, $\sigma \in \mathsf{SL}_2(\mathbb{R})$ and thus is also holomorphic.)

Exercise 6. Suppose that Γ, Γ' are Fuchsian groups and Γ' is a subgroup of Γ.

(i) Show that the natural map $X_\Gamma \to X_{\Gamma'}$ is holomorphic.
(ii) Show that if Γ' is of finite index in Γ then they have the same cusps; and hence, $\mathfrak{H}_\Gamma = \mathfrak{H}_{\Gamma'}$.
(iii) Deduce from Siegel's theorem that if Γ' is a subgroup of Γ of finite index then Γ is of the first kind if and only if Γ' is. Show that in that case the degree of the holomorphic map $X_\Gamma \to X_{\Gamma'}$ is exactly the index $[\overline{\Gamma} : \overline{\Gamma'}]$ where

$$\overline{\Gamma} := \Gamma/(\Gamma \cap \{\pm I\})$$

is the projectivization of Γ.

(Hint: the volumes differs by a factor of $[\overline{\Gamma} : \overline{\Gamma'}]$.)

Exercise 7. This exercise is to illustrate Fourier expansion without real analysis. We consider the case where the discrete subgroup $\Gamma = \langle T_h \rangle$

is the cyclic group generated by translation matrix $T_h := \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ for some $h > 0$.

(i) Show that Γ has only one cusp ∞.

(ii) Imagine what the topological space Y_Γ and X_Γ should look like.

(iii) Prove that the extended exponential map

$$q_h : \mathfrak{H}_\Gamma \to \mathfrak{D} = \mathfrak{D}_1$$

given by

$$q_h(z) = \begin{cases} e^{2\pi i z/h} & \text{if } z \in \mathfrak{H}, \\ 0 & \text{if } z = \infty. \end{cases}$$

induces a bi-holomorphic isomorphism between the Riemann surface X_Γ and $\mathfrak{D}$.

(iv) As a result, deduce that if $f : \mathfrak{H} \to \mathbb{C}$ is a holomorphic function that is invariant under Γ i.e. is periodic of period h then the function $g = f \circ q_h^{-1}$ is holomorphic on $\mathfrak{D}^* = \mathfrak{D} - \{0\}$. Thus let

$$g = \sum_{n=-\infty}^{\infty} a_n w^n$$

be the Laurent expansion of g at 0. Then $f = g \circ q_h$ has Fourier expansion

$$f = \sum_{n=-\infty}^{\infty} a_n q_h^n.$$

The meromorphicity or holomorphicity of f at ∞ is precisely the behavior of the singularity $0 \in D$ of the function g, namely whether it is a removable singularity or a pole of g.

(v) Under the above interpretation, derive equivalent conditions for "holomorphic at ∞" using Riemann's theorem on removable singularity[1].

1.3. **Fundamental Domain.** As X_Γ is the quotient of $\mathfrak{H}_\Gamma$, it is useful for computation and intuition to have a good subset of representatives for points in X_Γ.

Definition 1.3. A connected domain $F \subset \mathfrak{H}$ is called a *fundamental domain* of Γ if

(i) $\mathfrak{H} = \bigcup \gamma F$;
(ii) $F = \overline{U}$ where U is the interior of F i.e. F is closed;
(iii) $\gamma U \cap U = \emptyset$ for any $\gamma \in \Gamma, \gamma \neq \pm I$.

The existence of fundamental domain was proved in Section 1.6 of [1]. The idea was similar to how one find fundamental domain of $\mathsf{SL}_2(\mathbb{Z})$; namely, somehow pick an optimal

[1]For instance, 0 is removable singularity $\iff$ $g(w)$ is bounded near 0; translate this to condition on f gives ...

representative. Except that in general, one has to use the geometry of $\mathfrak{H}$: Pick a non-elliptic point $z_0 \in \mathfrak{H}$ and add geodesics starting from z_0. More precisely,

$$F := \left\{ z \in \mathfrak{H} \,\middle|\, \begin{array}{l} d(z, z_0) \leq d(z, \gamma z_0) \\ \text{for all } \gamma \in \Gamma \end{array} \right\}$$

is a fundamental domain of Γ. Here, d denotes Poincare metric (hyperbolic distance) on $\mathfrak{H}$. Basically, amongst all representatives, we choose the one closest to z_0.

This construction has several other properties:

(i) geodesic between any two points in F lies in F;
(ii) for any $\gamma \in \Gamma - \{\pm I\}$, $L_\gamma := F \cap \gamma F$ is contained in $C_\gamma := \{z | d(z, z_0) = d(z, \gamma z_0)\}$ and if $L_\gamma \neq \emptyset$ then it is a singleton or a geodesic;

- Let me recall that a geodesic on $\mathfrak{H}$ is either a vertical line or a semi-circle orthogonal to the real axis.
- We call L_γ a *side* of F if $L_\gamma \neq \emptyset$ and not a singleton. Note that the sides make up the boundary of F.
- If L, L' are two sides and $L \cap L' \neq \emptyset$ then the point in the singleton set $L \cap L'$ is called a *vertex* of F.
- If L has no end, we call the intersection of L and $\mathbb{R} \cup \{\infty\}$ a *vertex* of F (and *end point* of L) on $\mathbb{R} \cup \{\infty\}$.

(iii) for any compact $M \subset \mathfrak{H}$, $\{\gamma \in \Gamma \mid M \cap \gamma F$ is finite;

(iv) if a vertex $x \in \mathbb{R} \cup \{\infty\}$ of F is an end of two distinct sides and x is fixed by a non-scalar element of Γ then x is a cusp of Γ.

(v) If Γ is of the first kind, then any vertex of F on $\mathbb{R} \cup \{\infty\}$ is a cusp of Γ and any cusp of Γ is equivalent to a vertex of F on $\mathbb{R} \cup \{\infty\}$.

One utility of a fundamental domain is that it makes it easy to pick or visualize a chart:

- If a point on X_Γ comes from a point in the interior of F, then we can take a disk in F and the chart is "identity" map. (That said, an interior point cannot be a cusp or elliptic point. For then we have $\Gamma_\tau \subset \{\pm I\}$ by definition of fundamental domain and so τ is not fixed by an elliptic or parabolic element.)
- If a point on a side (boundary) is ordinary, then it will be identified with another point and basically, it is just two half-disks joining together.
- In the case we are interested in i.e. Γ is of the first kind, cusps are just vertices of the fundamental domain on $\mathbb{R} \cup \{\infty\}$.

Exercise 8. A fundamenetal domain for $\Gamma = \mathsf{SL}_2(\mathbb{Z})$ is well-known to be

$$F = \left\{ z \in \mathfrak{H}; |z| \geq 1 \text{ and } -\frac{1}{2} \leq \Re(z) \leq \frac{1}{2} \right\}.$$

(i) Verify that fact.
(ii) From this fundamental domain, what is Y_Γ and X_Γ topologically?
(iii) What are the sides of F and their endpoints? What are the vertices of F?

2. Modular function of Weight 0 are functions on X_Γ

A modular function f of weight 0 and level Γ satisfies the functional equation

$$f(\gamma\tau) = f(\tau)$$

for every $\gamma \in \Gamma$. Thus, it induces a meromorphic function

$$f : X_\Gamma \to \mathbb{C}$$

where we send $P \mapsto f(\tau)$ if $P = \pi(\tau)$ and $\tau \in \mathfrak{H}$ and if $P = \pi(x)$ is a cusp, we send P to the value of f at the cusp i.e. the a_0 coefficient in the Fourier expansion of f at the cusp x (or ∞ if f is not holomorphic at the cusp x). We need to show that this is well-defined.

Exercise 9. Show that the value of f at the cusp x is independent of the choice of the matrix σ that sends x to ∞.

Proof. It is easy to reduce the problem to x simply be ∞. Then a matrix in $\mathsf{SL}_2(\mathbb{R})$ fixing ∞ is of the form $\sigma = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ for $a \in \mathbb{R}^\times, b \in \mathbb{R}$. We have to show that

$$\lim_{z \to i\infty} f(z) = \lim_{z \to i\infty} f(\sigma z).$$

One has

$$\lim_{z\to i\infty} f(\sigma z) = \lim_{z\to i\infty} f\left(\frac{az+b}{a^{-1}}\right)$$
$$= \lim_{z\to i\infty} f(a(az+b))$$
$$= \lim_{z\to i\infty} f(a^2 z+b)$$

Note that as $z \mapsto +i\infty$, we also have $a^2z + b \to +i\infty$ because $a^2 > 0$ (albeit on a different path). Thus, we obtain the equality of limits. □

Conversely, a meromorphic function on X_Γ gives rise to a modular function of weight 0 by composition with the quotient map $\pi : \mathfrak{H} \to X_\Gamma$.

Thus, there is a one-to-one bijection between the field of meromorphic function on X_Γ and modular function of weight 0 and level Γ:

$$K(X_\Gamma) \cong A_0(X_\Gamma).$$

A modular form of weight 0 can similarly be interpreted as a holomorphic modular function on X_Γ i.e.

$$\mathcal{O}(X_\Gamma) \cong M_0(X_\Gamma).$$

The concept of modular form of weight 0 is not very interesting; as the exercise shows.

Exercise 10. Prove that if X_Γ is compact then there is no non-constant modular form of weight 0.

Proof. A meromorphic function has the same number of poles as zeros. A non-constant holomorphic function between compact Riemann surface has to be surjective; in particular, non-constant $X_\Gamma \to \mathbb{P}^1_\mathbb{C}$ must obtain some zero and pole. □

3. Modular functions and modular forms of even weight

Suppose that f is a modular function of weight 2. I shall show that

$$\omega := (U_P, z_P, \varphi_P)_{P \in X_\Gamma}$$

where

$$\varphi_P(Q) := f(\pi^{-1}(Q)) \left(\left. \frac{d(z_P \circ \pi)}{dz} \right|_{\pi^{-1}(Q)} \right)^{-1}$$
$$= \begin{cases} f(z_P^{-1}(Q)) & \text{if } P \in \pi(\mathfrak{H}), \\ f(z_P^{-1}(Q)) \left(\frac{d(q_h \circ \sigma)}{dz} \right)^{-1} & \text{otherwise} \end{cases}$$

is a (meromorphic) differential of degree 1 on X_Γ. The derivative is simply the derivative appearing above is simply of the function on the lower row

$$\begin{array}{ccccc} \mathfrak{H}_\Gamma & \xrightarrow{\pi} & X_\Gamma & & \\ & & & \searrow^{z_P} & \\ V_{\pi^{-1}(P)} & \longleftrightarrow & U_P & \longrightarrow & W_P \end{array}$$

where $\pi^{-1}(P)$ is the chosen image of P in the chart and $V_{\pi^{-1}(P)} \subset \mathfrak{H}_\Gamma$ is the chosen open subset in the chart. Basically, we want to push forward the differential form $f(z)dz$ on $\mathfrak{H}$ to X_Γ; but unfortunately, forms can only be pull-backed.

By definition, we have to check that for any two points $P, Q \in X_\Gamma$ and any point $T \in U_P \cap U_Q$ then

$$\varphi_P(T) \left(\frac{dz_P}{dz_Q} \right) (T) = \varphi_Q(T). \tag{3.1}$$

Observe that (3) is totally local on T. This allows us to show (3) by connecting it via U_T; namely, if it is true for any P, Q, T then it must be true when we apply it to $(P, Q, T) = (P, T, T)$ and when $(P, Q, T) = (T, Q, T)$ for if $T \in U_P \cap U_Q$ then $T \in U_P \cap U_T$ and $T \in U_Q \cap U_T$:

$$\varphi_P(T) \left(\frac{dz_P}{dz_T} \right) (T) = \varphi_T(T) \tag{3.2}$$

$$\varphi_T(T) \left(\frac{dz_T}{dz_Q} \right) (T) = \varphi_Q(T). \tag{3.3}$$

so then multiplying and cancelling $\varphi_T(T)$ on both sides[2] we get

$$\varphi_P(T) \left(\frac{dz_P}{dz_T} \right) (T) \left(\frac{dz_T}{dz_Q} \right) (T) = \varphi_Q(T)$$

and we see a familiar instance of the chain rule yield (3).

So the upshot is that we only need to show when one of the point lies in the local chart at

[2]This should be ok because the zero are discrete and if two meromorphic functions agree on a dense set then they are identical.

the other point, say $Q \in U_P$, and at T being the former point i.e. $T = Q$, namely

$$\varphi_P(Q)\left(\frac{dz_P}{dz_Q}\right)(Q) = \varphi_Q(Q). \tag{3.4}$$

Recall $\tau_P, \tau_Q, V_\tau, V_\rho \subset \mathfrak{H}_\Gamma$ be the chosen representatives of P and Q and the neighborhood in the chosen chart. Note that $U_P = \pi(V_{\tau_P})$ and since $Q \in U_P$, we can find $\tau'_Q \in V_{\tau_P}$ that also represents Q; thus, they must be related by an element $\gamma \in \Gamma$, say $\tau'_Q = \gamma\tau_Q$.

If P is not a cusp, the transition from $V_{\tau_Q} \to V_{\tau_P}$ is simply given by the matrix γ; and the equation (3.4) is simply assertion that

$$f(\gamma\tau_Q)\left.\frac{d\gamma}{dz}\right|_{z=\tau_Q} = f(\tau_Q)$$

which is nothing but the functional equation satisfied by modular functions of weight 2: Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which is identified with the holomorphic function $z \mapsto \frac{az+b}{cz+d}$ then

$$\begin{aligned}\gamma'(z) &= \frac{a(cz+d)-c(az+b)}{(cz+d)^2} \\ &= \frac{ad-bc}{(cz+d)^2} \\ &= \frac{1}{(cz+d)^2} \text{ if } \gamma \in \mathsf{SL}_2(\mathbb{R})\end{aligned}$$

If P is a cusp, the reasoning is similar and the factor $\left(\frac{d(q_h \circ \sigma)}{dz}\right)^{-1}$ in φ_P cancel the effect of the transition map.

We have thus verified that a modular function of weight 2 is essentially a differential of degree 1 on the modular curve X_Γ.

The same argument should work for weight $2k$: A modular function of weight $2k$ gives rise to a differential of degree k on X_Γ. Thus, from the general theory of compact Riemann surfaces (i.e. assume compactness of X_Γ), we know that the modular functions of degree $2k$

are 1-dimensional vector space over the function field $\mathcal{K}(X_\Gamma)$ of X_Γ.

It is an easy check that the subspace of forms (i.e. the modular functions that are holomorphic) of weight $2k$ whose order of vanishing at all cusps $\geq k$ are in correspondence with holomorphic differential of degree k. In particular,

$$S_2(\Gamma) \cong \Omega^1(X_\Gamma).$$

We are going to exploit this geometric interpretation to compute the dimension of the space of modular forms of even weight.

References

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