Modular Forms and Complex Analysis I

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Our standard references are Miyake [4], Lang [3], Koblitz [2], Shimura [5]. I find that each of these books are good for different purposes. Shimura explains fairly well the computation of dimension of the space of modular forms. Miyake has more details on the Riemann surfaces. Koblitz is more hand-on, have lots of exercises. Lang's book is harder to read (especially on notations) but provides a good account of the field of modular functions, complex multiplication, etc.

The goal of this first note is to define and give examples of modular forms. I am going to assume basic knowledge of complex analysis such as [1].

1. Classical modular forms

Let $\mathfrak{H}$ denote the upper half plane.

Definition 1.1. Let $\Gamma \subset \mathsf{SL}_2(\mathbb{R})$ be a discrete subgroup[1] and $k \in \mathbb{Z}$. A holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ is called a *modular form of level* Γ *and weight* k if $f|_k\gamma = f$ for all $\gamma \in \Gamma$ and that f is *holomorphic at the cusps*.

A modular form f is called a *cusp form* if it vanishes at every cusps.

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[1] Recall that SL_2 is an algebraic group and $\mathsf{SL}_2(\mathbb{R})$ has natural topology via embedding to $\mathbb{R}^4$. A discrete subgroup of $\mathsf{SL}_2(\mathbb{R})$ is normally called a *Fuchsian group*.

Here, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the $|_k$ action (read: the slash k action) is defined by

$$(f|_k\gamma)(z) := (\det\gamma)^{k/2} j(\gamma, z)^{-k} f\left(\frac{az+b}{cz+d}\right),$$

and

$$j(\gamma, z) := cz + d$$

is called the *factor of automorphy*.

So roughly speaking, a modular form is simply a function satisfying a certain collection of functional equations. More general is the concept of a modular function which essentially replace "holomorphic" by "meromorphic" in definition Definition 1.1.

Definition 1.2. A meromorphic function $f : \mathfrak{H} \to \mathbb{C}$ is called a *modular function of level* Γ *and weight* k if $f|_k\gamma = f$ for all $\gamma \in \Gamma$ and that f is *meromorphic at the cusps.*

We shall explain the "holomorphic (and meromorphic) at the cusp" and the "vanish at the cusp (more generally the value of a modular form at a cusp)" later after we discuss Fourier expansion, but it should corresponds to the "moderate growth condition" in the definition of automorphic forms. In particular, it is equivalent to $f(z) = \mathrm{O}(\Im(z)^{-v})$ as $\Im(z) \to 0$ uniformly with respect to $\Re(z)$ for some $v \in \mathbb{R}^+$ and if we can choose $v < k$ then f is a cusp form. (See Theorem 2.1.4 in [4].) Alternatively, it is also equivalent to

$$\lim_{z \to i\infty} (f|_k \gamma)(z) < \infty$$

for all $\gamma \in \Gamma$. We can use these equivalent condition as definition but they are not very intuitive and does not explain the terminology well.

Notation. We denote $M_k(\Gamma)$ (and $A_k(\Gamma)$) for the set of all modular forms (modular functions, resp.) of weight k and level Γ.

Exercise 1. Verify that the slash k action really defines a (right) group action of $\mathsf{GL}_2(\mathbb{R})^+$ on the set of all functions $f : \mathfrak{H} \to \mathbb{C}$. In other words, $f|_k\, I_2 = f$ and $(f|_k\gamma)|_k\sigma = f|_k(\gamma\sigma)$ for any such function where I_2 is 2×2 identity matrix.

Notation. For future use, let me also denote by $M'(\Gamma)$ and $A'(\Gamma)$ the set of holomorphic (and meromorphic, resp) functions $\mathfrak{H} \to \mathbb{C}$ satisfying the functional equation $f|_k\gamma = f$. Then $M(\Gamma) \subset M'(\Gamma)$ is the subset of those satisfying the extra holomorphy condition at the cusps.

Exercise 2. Let $\Box$ be one of M', A'.

(i) Show that $\Box_k(\Gamma)$ is a $\mathbb{C}$-vector space and that if $f \in \Box_k(\Gamma)$ and $g \in \Box_\ell(\Gamma)$ then $fg \in \Box_{k+l}(\Gamma)$.

(ii) Assuming Γ is Fuchsian group of the first kind. Show that we have graded rings

$$\Box(\Gamma) := \bigoplus_{k=-\infty}^{\infty} \Box_k(\Gamma).$$

Note that this means that

$$\Box_k(\Gamma) \cap \Box_\ell(\Gamma) = \{0\}$$

for $k \neq \ell$. (Lemma 2.1.1 in [4].)

(iii) Show that if $f \in \Box_k(\Gamma)$ then $f|_k\alpha \in \Box_k(\alpha^{-1}\Gamma\alpha)$ for any $\alpha \in \mathsf{GL}_2^+(\mathbb{R})$. Note in particular that the function $f|_k\alpha$ remains being holomorphic (meromorphic) should the original function is.

(In the future, do this for $\Box = M, A$.)

2. Congruence subgroups

In number theory, one typically concerns with the following Hecke-type discrete subgroups

$$\Gamma(N) := \ker(\mathsf{SL}_2(\mathbb{Z}) \to \mathsf{SL}_2(\mathbb{Z}/N\mathbb{Z}))$$
$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) | c \equiv 0 \bmod N \right\}$$
$$\Gamma_1(N) := \ker(\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times)$$

where the map $\mathsf{SL}_2(\mathbb{Z}) \to \mathsf{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is entry-by-entry reduction mod N and the map

$$\Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$$

sends a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \bmod N$.

Exercise 3. Check that these are *surjective* group homomorphisms.

The above definition makes it clear that $\Gamma(N)$ is normal subgroup of $\Gamma(1) = \mathsf{SL}_2(\mathbb{Z})$;

$\Gamma_1(N)$ is normal subgroup of $\Gamma_0(N)$ and that

$$\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^*.$$

Definition 2.1. A discrete subgroup containing $\Gamma(N)$ for some integer N is called a *congruence subgroup*.

A very first example of modular forms are *Eisenstein series*. For $k \geq 4$ *even*, let

$$G_k(z) := \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} (mz+n)^{-k}.$$

Remark 2.2. Technically, the notation

$$\sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2}$$

is not well-defined because it did not specify the order of summation; without such, we cannot form partial sum. Fortunately, if $k \geq 4$, the convergence is absolute (Lemma 4.1.6 of [4]) so the order does not matter. One good ordering is to go counter clockwise on squares layers of "radius" R as $R = 1, 2, ...,$ i.e. let

$$L_R := \left\{ (m,n) \;\middle|\; \begin{array}{l} m, n \in \mathbb{Z} \\ |m|, |n| \leq R \\ (m^2 - R^2)(n^2 - R^2) = 0 \end{array} \right\}$$

and define

$$G_k(z) := \sum_{R=1}^{\infty} \sum_{(m,n)\in L_R} (mz+n)^{-k}.$$

Note that L_R is a finite set; in fact, $|L_R| = 8R$ so the inner sum is a finite sum.

For every $z \in \mathfrak{H}$, let $P_{R,z}$ to be the parallelogram (4 line segments) with vertices $Rz + R, -Rz + R, -Rz - R, Rz - R$ and let $r(z)$ be the minimal distance from 0 to $P_{1,z}$. It follows that if $(m,n) \in L_R$ then $|mz + n| \geq r(z)R$

since $P_{R,z} \supset \{mz+n|(m,n) \in L_R\}$. And so

$$\begin{aligned}\sum_{R=1}^{\infty} \sum_{(m,n)\in L_R} |mz+n|^{-k} &\leq \sum_{R=1}^{\infty} |L_R| \times (r(z)R)^{-k} \\ &\leq r(z)^{-k} \sum_{R=1}^{\infty} 8R^{1-k} \\ &\leq 8r(z)^{-k}\zeta(k-1).\end{aligned}$$

Since $r(z) : \mathfrak{H} \to \mathbb{R}_+$ is continuous, we also find that $G_k(z)$ converges uniformly on compact subset.

Exercise 4. Let $(a_n)_n$ be sequence of real numbers.

(i) Show that if $\sum a_n$ converges absolutely i.e. $\sum |a_n| < \infty$ then $\sum a_{\sigma(n)} = \sum a_n$ for all permutation $\sigma : \mathbb{Z}^+ \to \mathbb{Z}^+$.

(ii) Show Riemann's result that if $\sum a_n$ converges conditionally then for any $r \in \mathbb{R} \cup \{\pm\infty\}$, there exists a rearrangement $\sigma : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $\sum a_{\sigma(n)} = r$.

Now it is easy to verify that G_k is a modular form of weight k and level $\Gamma(1) = \mathsf{SL}_2(\mathbb{Z})$.

Exercise 5. Verify the above statement. What is the reason it only works for $k \geq 4$? In other words, what breaks down when $k = 2$? (*Answer*: Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$. One has

$$\begin{aligned} & G_k\left(\frac{az+b}{cz+d}\right) \\ & = \sum \left(m\frac{az+b}{cz+d} + n\right)^{-k} \\ & = (cz+d)^k \sum (m(az+b) + n(cz+d))^{-k} \\ & = j(\gamma, z)^k \sum ((ma+nc)z + (mb+nd))^{-k} \end{aligned}$$

Since $\gamma \in \Gamma(1)$, we see that $\gamma\mathbb{Z}^2 = \mathbb{Z}^2$; in other words, as (m, n) runs over all pairs of integers, $(ma + nc, mb + nd)$ also. Hence, the last sum is a permutation of the sum in the definition of E_k. Unfortunately, we cannot conclude that the permuted sum equals E_k unless the sum E_k *converges absolutely*; which only does for $k \geq 4$!

See Koblitz's Proposition for the *proper definition* of E_2 and the functional equation satisfied by it.)

Remark 2.3. As a remark, it is well-known that $\mathsf{SL}_2(\mathbb{Z})$ is generated by the two matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{(reflection matrix)}$$

and

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{(translation matrix)}$$

i.e. any matrix in $\mathsf{SL}_2(\mathbb{Z})$ can be written as a word in S and T, namely $S^{i_1}T^{j_1}S^{i_2}...S^{i_n}T^{i_n}$ with $i_k, j_k \in \mathbb{Z}$. As a result, to verify that f is a modular form of level $\Gamma(1)$, we only need to verify the functional equation where $\gamma = S$ and $\gamma = T$.

Other congruence subgroups are also finitely generated. See [6].

3. Linear fractional transformations

In this section, we are going to state classification of linear fractional transformation in $\mathsf{GL}_2(\mathbb{C})$ in [5]. ([4] only did it for $\mathsf{GL}_2(\mathbb{R})$ which is all we need but there's no harm doing more.) Observe that a matrix $\sigma \in \mathsf{GL}_2(\mathbb{C})$ is conjugate to its Jordan canonical form which looks either like $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ $(z \mapsto \lambda^{-1})$ or like $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ $(z \mapsto \lambda/\mu z)$ where $\lambda \neq \mu$. Then $\sigma \neq I_2$ is called

(i) *parabolic* in the first case $\iff$ σ has exactly one fixed point ∞;
(ii) *elliptic* if $|\lambda/\mu| = 1$;
(iii) *hyperbolic* if $\lambda/\mu \in \mathbb{R}^+$;

(iv) *loxodromic* otherwise.

In case (ii)-(iv) σ has two fixed points, namely 0 and ∞. (Fixed points are all considered on $\mathbb{P}^1_{\mathbb{C}}$.)

If $\sigma \in \mathsf{SL}_2(\mathbb{C})$, one simplifies to

(i) $\operatorname{Tr}(\sigma) = \pm 2$;
(ii) $\operatorname{Tr}(\sigma) \in (-2, 2) \subset \mathbb{R}$;
(iii) $\operatorname{Tr}(\sigma) \in (-\infty, -2) \cup (2, +\infty) \subset \mathbb{R}$; and
(iv) $\operatorname{Tr}(\sigma) \notin \mathbb{R}$,

respectively.

When $\sigma \in \mathsf{SL}_2(\mathbb{R})$, (iv) cannot happen and the rests are equivalent to

(i) σ has one fixed point on $\mathbb{R} \cup \{\infty\}$;
(ii) σ has a fixed point $z \in \mathfrak{H}$ and $\overline{z}$ is the other fixed point;
(iii) σ has two fixed points on $\mathbb{R} \cup \{\infty\}$.

4. Cusps and Elliptic Points

Fix a discrete subgroup Γ.

Definition 4.1. A point x of $\mathbb{P}^1_{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ is called a *cusp* of Γ if it is fixed by a parabolic element of Γ.

Similarly, a point $z \in \mathfrak{H}$ is called an *elliptic point* of Γ if it is fixed by an elliptic element of Γ.

There is also the notion of *hyperbolic point* (you can guess what this should be) but we don't need that notion.

The isotropy group of elliptic points and cusps can be easily determined. Let me state Theorem 1.5.4 in [4] (san hyperbolic point statement) which I normally referred to as "structure theorem for isotropy group".

Theorem 4.2. *(i) If z is a cusp then Γ_z is a finite cyclic group.*

(ii) If x is a cusp then every element of Γ_x is parabolic and $\Gamma_x/\Gamma \cap \{\pm I\} \cong \mathbb{Z}$. In fact, let $\sigma \in \mathsf{SL}_2(\mathbb{R})$ be such that $\sigma x =$

∞. *Then*

$$\pm\sigma\Gamma_x\sigma^{-1} = \left\{ \pm \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}^m | m \in \mathbb{Z} \right\}$$

for some $h \in \mathbb{R}_+$.

The real number h appearing Theorem 4.2 in is called the *width of the cusp* x. It should be independent of σ chosen.

Now let assume $-I \notin \Gamma$, x a cusp and $\sigma x = \infty$ as in the theorem. In this case $\Gamma \cap \{\pm I\} = \{I\}$ so Theorem 4.2 shows that $\Gamma_x \cong \mathbb{Z}$ so $\sigma\Gamma_x\sigma^{-1}$ contains either $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & h \\ 0 & -1 \end{pmatrix}$ (exclusive[2]). We call a cusp *regular* or *irregular* correspondingly.

Exercise 6. Show that the regularity of the cusp is independent of the σ chosen. (Lemma 1.5.6 in [4].)

[2]In case $-I \in \Gamma$, it contains both.

Exercise 7. If $\Gamma' \subset \Gamma$ is a subgroup of finite index then the sets of cusps of Γ' and Γ coincide.

Exercise 8. Determine the cusps of $\Gamma(1)$ and $\Gamma_{\square}(N)$ in general. (Answer: $\mathbb{P}^1_{\mathbb{Q}}$.)

5. Fourier expansion

Now we are ready to talk about Fourier expansion at the cusps.

Exercise 9. Recall Fourier expansion of a real periodic function: Let $f : \mathbb{R} \to \mathbb{R}$ be periodic of period h i.e. $f(x+h) = f(x)$ for all $x \in \mathbb{R}$ and suppose that f is integrable on $[a, a+p]$. Then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{p}\right) + b_n \sin\left(\frac{2\pi n x}{p}\right)$$

for some real numbers a_n, b_n; more precisely

$$a_n = \frac{2}{p}\int_a^{a+p} f(x)\cos\left(\frac{2\pi nx}{p}\right)dx,$$
$$b_n = \frac{2}{p}\int_a^{a+p} f(x)\sin\left(\frac{2\pi nx}{p}\right)dx.$$

(i) Discuss the convergence of the series: Suppose a_n, b_n are given. For what range of x does the series converge? (One thing is for sure: Since $|\sin|, |\cos| \leq 1$, it is clear that if $\sum |a_n| + |b_n|$ converges then the series converge for every x.)

(ii) Prove that the function given by Fourier series $g(x) = \frac{a_0}{2} + ...$ equals $f(x)$ at the point of continuity and that average of left/right limits at the point of discontinuity.

As a remark, Fourier series is in general related to decomposition of $L^2(\mathbb{R}/\mathbb{Z})$.

Let Γ be a fixed discrete subgroup, x be a cusp of Γ, $f \in M_k'(\Gamma)$ and σ be such that $\sigma x = \infty$. In exercise, we have seen that

$$f|_k\sigma^{-1} \in M_k'(\sigma\Gamma\sigma^{-1})$$

and so

$$f|_k\sigma^{-1} \in M_k'(\sigma\Gamma_x\sigma^{-1})$$

as $\Gamma_x \subset \Gamma$.

- If x is a regular cusp, then $\sigma\Gamma_x\sigma^{-1}$ contains $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ and the functional equation for such matrix read

  $$(f|_k\sigma^{-1})(z+h) = (f|_k\sigma^{-1})(z)$$

  In other words, $f|_k\sigma^{-1}$ is periodic of period h. Together with $f|_k\sigma^{-1}$ being holomorphic, we deduce that $f|_k\sigma^{-1}$ can be expressed as a function $g(e^{2\pi iz/h})$ for some function $g(z)$ holomorphic away from 0; in other words, it can be expressed as a Laurent series in $q_h := e^{2\pi iz/h}$, namely

$$f|_k\sigma^{-1} = \sum_{n=-\infty}^{\infty} a_n q_h^n.$$

- If x is an irregular cusp then we similarly find that $\sigma\Gamma_x\sigma^{-1}$ contains $\begin{pmatrix} -1 & h \\ 0 & -1 \end{pmatrix}$ and so we find that

  $$(-1)^k(f|_k\sigma^{-1})(z-h) = (f|_k\sigma^{-1})(z)$$

  so $f|_k\sigma^{-1}$ is either periodic of period $2h$ or of period h. The rest of the above reasoning applies; that is to say, $f|_k\sigma^{-1}$ can be expressed as Laurent series in q_{2h}, namely

  $$f|_k\sigma^{-1} = \sum_{n=-\infty}^{\infty} a_n q_{2h}^n.$$

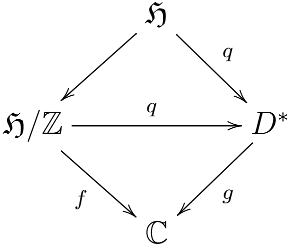
Such expansion is called *Fourier expansion* of f and the numbers a_n are naturally called *Fourier coefficients*.

Definition 5.1. A function $f \in M'_k(\Gamma)$ is said to be holomorphic at the cusp x if the Fourier coefficients $a_n = 0$ for all $n < 0$.

If f is holomorphic at the cusp x, we can define $f(x) := a_0$ so that the function f extends to a function $\mathfrak{H} \cup \{\text{cusps}\}$. The problem is that Fourier expansion at x potentially depends on a choice of σ such that $\sigma x = \infty$.

Remark 5.2. Why does a periodic holomorphic function has Fourier expansion? We can use either the real Fourier expansion theory (in the exercise above) or the Riemann surface theory (without integrability requirement) to explain it.

In other words, the "holomorphy" is a really strong condition. The idea is that a periodic holomorphic function f on $\mathfrak{H}$ (say, of period 1) gives rise to a holomorphic function (also denoted by f) on the quotient $\mathfrak{H}/\mathbb{Z}$ (where $\mathbb{Z}$ acts on $\mathfrak{H}$ by $[n] \cdot z = z + n$) which can be given a natural Riemann surface structure. Then the exponential map $q : \mathfrak{H} \to D^*; z \mapsto e^{2piiz}$ gives rise to bi-holomorphism between $\mathfrak{H}/\mathbb{Z}$ and D^*, where $D^* = \{|z| < 1, z \neq 0\}$.



Thus, from the periodic function f, we get a holomorphic function $g = f \circ q^{-1}$ on D^*.

Now the Fourier expansion capture the behavior of g at 0, which could be a removable singularity, a pole, or an essential singularity. In any case, g has Laurent series expansion at 0

$$g(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

whence $f = g \circ q$ has q-expansion

$$g(z) = \sum_{n=-\infty}^{\infty} a_n q^n.$$

To this end, an alternative method to define holomorphy at a cusp is to say that the corresponding function g has a removable singularity at zero.

Note that the value of a_0 is precisely the value one should assign at 0 to extend g to a holomorphic function on the whole of D.

Exercise 10. Show that f is holomorphic at a cusp x if and only if the limit

$$\lim_{z \to i\infty} (f|_k \sigma^{-1})(z)$$

exists and is finite. (*Answer*: Follow from the remark. The function g has removable singularity at zero precisely when $\lim_{z \to 0} g(z)$ exists and is finite by Riemann's theorem on removable singularity.)

Exercise 11. Derive the formula for the Fourier coefficients of a modular form whose level contains $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$. Make explicit the assumptions. (*Answer:* From the real Fourier expansion:

$$a_n = \frac{1}{h} \int_{z_0}^{z_0+h} f(z) e^{-2\pi i n z/h} dz$$

for any choice of $z_0 \in \mathfrak{H}$. There is an alternative interpretation of this formula: it is just Taylor expansion + Cauchy's integral formula for the function $g(w)$ such that $f = g \circ e^{2\pi i z/h}$.)

Exercise 12. We obtained the integral formula for Fourier coefficients of a modular form in previous exercise. Unfortunately, such a formula is usually difficult to compute due to an integral. Find an alternative analytic method to compute the Fourier coefficient. (*Idea:* The Fourier coefficients of $f(z) = g(q)$

are the coefficients in the power series expansion of $g(z)$, which can be obtained from Taylor series i.e. $a_n = \frac{1}{n!}\frac{d^n g}{dz^n}|_{z=0}$. First of all,

$$a_0 = g(0) = f(i\infty) = \lim_{z\to i\infty} f(z)$$

Differentiating both sides of

$$f(z) = \sum_{n=0}^{\infty} a_n q^n$$

with respect to z (note that $dq = 2\pi i\ q\ dz$), we can also see that

$$f'(z) = \sum_{n=1}^{\infty} n a_n q^{n-1}(2\pi i q) = 2\pi i \sum_{n=1}^{\infty} n a_n q^n$$

and so

$$a_1 = \lim_{z\to i\infty} \frac{f'(z)}{2\pi i q}.$$

Continue for higher coefficients by taking derivatives and we find that

$$a_n = \frac{1}{n!} \lim_{z \to i\infty} \left(\frac{1}{2\pi i q} \frac{d}{dz} \right)^n f$$

i.e. apply the differential operator $\frac{1}{2\pi i q} \frac{d}{dz}$ for n times, starting with the function f; and take limit of the resulting function as $z \to i\infty$. To prove this formally, just note by chain rule that

$$\begin{aligned} a_n &= \frac{1}{n!} \frac{d^n g}{dq^n} \\ &= \frac{1}{n!} \left(\frac{d}{dz} \frac{dz}{dq} \right)^n f \\ &= \frac{1}{n!} \left(\frac{d}{dz} \frac{1}{2\pi i q} \right)^n f \end{aligned}$$

and we have to take limit since $q = 0$ is not in the domain.)

Exercise 13. Show that holomorphy at a cusp x only depends on Γ coset i.e. if f is holomorphic at x then it is holomorphic at γx for all $\gamma \in \Gamma$.

Exercise 14. Find the Fourier expansion of Eisenstein series $G_k(z)$ at the cusp ∞. (*Answer*: Using previous exercise, suppose that $G_k(z) = \sum_{n=0}^{\infty} a_n q^n$. We find that

$$\begin{aligned}
a_0 &= \lim_{z \to i\infty} G_k(z) \\
&= \lim_{z \to i\infty} \sum_{m,n} (mz+n)^{-k} \\
&= \sum_{m,n} \underbrace{\lim_{z \to i\infty} (mz+n)^{-k}}_{0 \text{ if } m \neq 0} \quad \text{(why switchable?)} \\
&= \sum_{n=-\infty, n \neq 0}^{\infty} n^{-k} \\
&= 2\zeta(k)
\end{aligned}$$

Note that this also shows that $G_k(z)$ is holomorphic at the cusp ∞ and completes the verification that $G_k(z)$ is a modular form for $\Gamma(1)$ as all cusp of $\Gamma(1)$ are equivalent to ∞.

We would like to use

$$\begin{aligned} a_1 &= \lim_{z\to i\infty} \frac{G_k'(z)}{2\pi i e^{2\pi i z}} \\ &= \lim_{z\to i\infty} \frac{\sum_{m,n}(-k)(mz+n)^{-k-1}m}{2\pi i e^{2\pi i z}} \\ &= \frac{-k}{2\pi i} \lim_{z\to i\infty} \frac{\sum_{m,n} m(mz+n)^{-k-1}}{e^{2\pi i z}} \end{aligned}$$

which doesn't seem to be easy to evaluate. It would be great if someone could come up with a treatment of these limits.

The final result is

$$G_k(z) = 2\zeta(k) + 2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

where $\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$. It could be obtained from the series development

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z+n} + \frac{1}{z-n} \right)$$

as well as

$$\pi \cot(\pi z) = \pi i \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} = \pi i (1 - 2 \sum q^n).$$

Differentiating these series for $\pi \cot(\pi z)$ for k-times yields the result.

After this exercise, one typically proceeds to define normalized Eisenstein series

$$\begin{aligned} E_k(z) &= \frac{1}{2\zeta(k)} G_k(z) \\ &= 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in \mathbb{Q}[[q]] \end{aligned}$$

from well-known zeta values at even integers. Here, B_k denotes the k-th Bernoulli number.

A better choice is to take

$$E_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in \mathbb{Q}[[q]]$$

to normalize the coefficient of q to 1. This is because (as we shall see) $E_k(z)$ is a Hecke eigenform and this normalization allows simple relation between Fourier coefficients and Hecke eigenvalues.

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