A SUMMARY OF HOMOLOGICAL ALGEBRA

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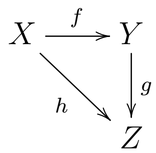
Abstract. I shall give a summary of homological algebra from categorical perspective. We assume the reader is familiar with the basic definition of **category** and **functor**. Most of the text is refined from wellknown sources.

1. First Glossary of Category Theory

Let $\mathcal{C}$ be a category.

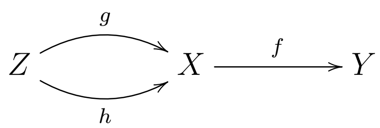
1.1. Commutative Diagram, Duality.

- **Diagram** in $\mathcal{C}$: A directed graph whose vertices are objects in $\mathcal{C}$ and edges are morphisms between the objects at the endpoints. (A diagram is not necessarily small i.e. its collection of vertices and edges need not be sets.)
- **Commutative diagram**: A diagram is said to "commute" if for any two paths from a vertex X to another vertex Y, the composed morphism is the same. For instance, commutativity of the diagram



  signifies that $h = g \circ f$ as morphisms in $\mathcal{C}$. Likewise, saying that the diagram

(*)



  commutes is equivalent to the equality $f \circ g = f \circ h$.
- **Duality**: Given a commutative diagram, one can reverse all arrows to get a commutative diagram in $\mathcal{C}^{\mathrm{op}}$, the opposite category of $\mathcal{C}$. A property in $\mathcal{C}$ defined using commutative diagram can be translated into an equivalent property a.k.a. the dual property in $\mathcal{C}^{\mathrm{op}}$ except with arrow reversed.

1.2. **Initial, Terminal, Zero objects.**

- **Initial object** (a.k.a. coterminal, universal object): An object $A \in \mathcal{C}$ such that for every object $X \in \mathcal{C}$, there exists precisely one morphism $A \to X$.
- **Terminal object** (or final): An object A such that for every object $X \in \mathcal{C}$, there exists precisely one morphism $X \to A$.

  Observe that the property of "being a terminal object" is the dual of the property of "being initial object": We simply reverse the arrow in the definition $A \to X$ to $A \leftarrow X$. Put another way, an object $A \in \mathcal{C}$ is terminal if and only if A is initial in $\mathcal{C}^{\text{op}}$.
- **Zero object**: An object that is both initial and terminal.

1.3. **Monomorphism, Epimorphism, Zeromorphism.**

- **Monomorphism**: A left-cancellative morphism i.e. $f : X \to Y$ such that for any object Z and any two morphisms $g, h : Z \to X$, if $f \circ g = f \circ h$ then $g = h$. In other words, there is no commutative diagram of the form (*) with $g \neq h$. Alternatively, for any Z, the induced map $f_* : \mathsf{Hom}(Z, X) \to \mathsf{Hom}(Z, Y)$ is *injective*.
- **Split monomorphism** (or a **section**): A left-invertible morphisms i.e. $f : X \to Y$ such that there exists $l : Y \to X$ such that $l \circ f = \mathsf{Id}_X$.
- **Epimorphism** and **split epimorphism**: Dual of momomorphism and split mono.
- **Constant morphism** (or left zero): A morphism f such that for any object W and any $g, h : Z \to X$, $f \circ g = f \circ h$ i.e. all diagram in $\mathcal{C}$ of the form (*) commutes.
- **Coconstant morphism** (or right zero): Dual of constant morphism.
- **Zeromorphism**: One that is both a constant morphism and a coconstant morphism.

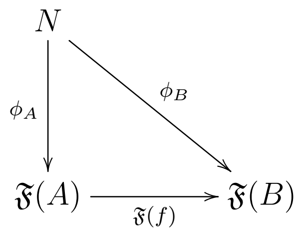
1.4. **Limit.**

- **Universal property**: Many notion in category theory (an object, a morphism, or pair of an object and a morphism) are defined using universal property. They indicate that the object we are defined are somehow the most general solution to a commutative diagram, in the sense that any other solution is a projection or subobject of the universal object in a canonical way.

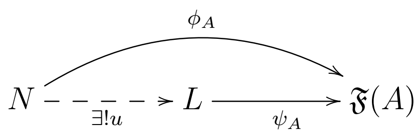
  Limit, product, equalizer, etc. are defined in this manner.

- **Limit**: Let $\mathcal{I}$ be a category (indexing category) and $\mathfrak{F} : \mathcal{I} \to \mathcal{C}$ be a functor which can be viewed as defining a diagram in $\mathcal{C}$: The vertices are $\mathfrak{F}(A)$ for each $A \in \mathcal{I}$ and edges are $\mathfrak{F}(f)$ for each morphism $f : A \to B$ in $\mathcal{I}$.

  An $\mathfrak{F}$-cone is an object $N \in \mathcal{C}$ together with a family of morphisms $\phi_A : N \to \mathfrak{F}(A)$ for each $A \in \mathcal{I}$ such that for any morphism $f : A \to B$ in $\mathcal{I}$, the diagram



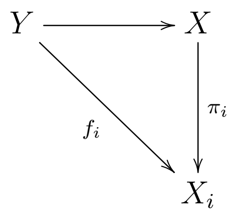
  commutes. The limit of $\mathfrak{F}$ is the universal cone i.e. the terminal object in the category of $\mathfrak{F}$-cone: It is the $\mathfrak{F}$-cone (L, ψ) such that for any $\mathfrak{F}$-cone (N, ϕ), there exists a *unique* morphism $u : N \to L$ such that $\phi_A = \psi_A \circ u$ for every $A \in \mathcal{I}$. In diagram:



- **Colimit**: Dual concept of limit.

To illustrate the concept of limit, let $\mathcal{C} = \mathbf{Ab}$ be the category of abelian groups and $\mathcal{I} = \mathbb{N}^{\geq}$ be the category of natural numbers (the objects are natural numbers $1, 2, ...$ and a unique morphism $i \to j$ is given for each $i \geq j$). To give a functor $\mathfrak{F} : \mathcal{I} \to \mathcal{C}$ is equivalent to giving a sequence of abelian groups $A_1, A_2, ...$ and group homomorphisms $f_{ij} : A_i \to A_j$ for each $i \geq j$ such that $f_{jk} \circ f_{ij} = f_{ik}$. The limit object in this case is precisely the familiar **inverse limit of groups** $\varprojlim A_i$.

- **Product**: Let $\{X_i\}_{i \in I}$ be a collection of objects in $\mathcal{C}$. The product of the family $\{X_i\}_{i \in I}$, denoted $\prod X_i$, is an object X together with a family of morphisms $\pi_i : X \to X_i$ such that for any other object Y and morphisms $f_i : Y \to X_i$, there exists a unique morphism $Y \to X$ such that the diagram



commutes.

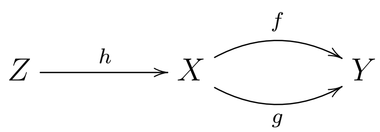
Observe that product is a special case of limit where the indexing category $\mathcal{I}$ is the so-called **discrete category** associated to the set I: the objects in $\mathcal{I}$ are elements of I and there is no other morphisms than the identity morphism $i \to i$ for every $i \in I$; and then we simply take the functor $\mathfrak{F}(i) = X_i$.

- **Coproduct**: Dual of product.
- **Biproduct**: An object together with projections and embedding that is both product and coproduct. Let $\{X_i\}_{i\in I}$ be a collection of objects. A biproduct is an object X together with $\pi_i : X \to X_i$ and $\kappa_i : X_i \to X$ such that (X, π_i) is a product and (X, κ_i) is a coproduct of the X_i's.

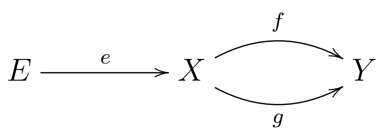
2. Second Glossary of Category Theory

Our goal is to introduce the notion of **abelian category** which is the cornerstone of homological algebra as it is the kind of categories in which **exact sequences** make sense.

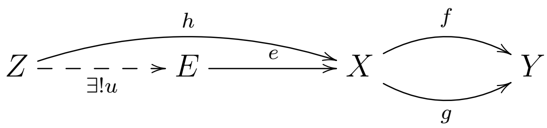
- **Equalizer** of two morphisms $f, g : X \to Y$: An object Z together with a morphism $h : Z \to X$ such that



commutes is said to *equalize* f and g. The equalizer is the universal pair of an object and a morphism for the property of "equalizing f and g". In other words, the equalizer of f and g consists of an object E together with a morphism $e : E \to X$ such that the diagram



commutes and if Z, h is another object and morphism that equalizes f, g then there exists a unique morphism $u : Z \to E$ such that



Note: Any equalizer is a monomorphism.

- **Coequalizer**: Dual of equalizer.
- **Kernel** of a morphism $f : X \to Y$: The equalizer between f and the zero morphism 0_{XY}. Concretely, a kernel of f is an object K together with a morphism $k : K \to X$ such that (i) $k \circ f$ is a zero morphism and (ii) for any other $K', k' : K' \to X$ such that $k' \circ f$ is a zero morphism, there is a unique morphism $u : K' \to K$ such that $k \circ u = k'$.
- **Normal monomorphism**: A monomorphism that is a kernel of some morphism.
- **Cokernel**: Dual of kernel.
- **Normal category**: Category where every monomorphism is normal.
- **Preadditive category**: Category where all Hom-sets are abelian groups and the composition of morphisms is bilinear.

  Note: Due to abelian group structures of Hom-set, preadditive category has zero morphisms 0_{XY} for every pair of objects X, Y. In general, as long as a category has a zero object, it has all zeromorphisms obtained by composition $X \to 0 \to Y$; so we can talk about kernel and cokernel.
- **Additive category**: A preadditive category where every finite set of objects has a biproduct.
- **Preabelian category**: An additive category where every morphism has both a kernel and a cokernel.
- **Abelian category**: A normal and co-normal preabelian category. Concretely, an abelian category is one which has a zero object, kernels, cokernels, all binary biproducts; whose every monomorphism is a kernel of some morphism, and whose every epimorphism is a cokernel of some morphism.
- **Injective object**: An object I such that for every mono $f : B \to C$, the pullback $f^* : \mathsf{Hom}(C, I) \to \mathsf{Hom}(B, I)$ is surjective. In diagram, for every $g : B \to I$, there exists a morphism to fill

in the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & B & \xrightarrow{f} & C \\ & & & \overset{g}{\searrow} & \downarrow \exists \\ & & & & I \end{array}$$

- **Projective object**: Dual of injective object.

$$\begin{array}{ccccc} 0 & \longleftarrow & B & \xleftarrow{f} & C \\ & & & \overset{g}{\nwarrow} & \uparrow \exists \\ & & & & P \end{array}$$

3. δ-functor and derived functor

Let $\mathcal{A}, \mathcal{B}$ be abelian categories.

- Exact cohomological δ-**functor** $\mathcal{A} \to \mathcal{B}$: A family $\mathsf{H} = (\mathsf{H}^n)_{n\in\mathbb{Z}}$ of functors $\mathcal{A} \to \mathcal{B}$ together with morphisms $\delta^n : \mathsf{H}^n(C) \to \mathsf{H}^{n+1}(A)$ for each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$ such that
  - Functorial: if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

    is commutative with short exact rows then

$$\begin{array}{ccc} \mathsf{H}^n(C) & \xrightarrow{\delta^n} & \mathsf{H}^{n+1}(A) \\ \downarrow & & \downarrow \\ \mathsf{H}^n(C') & \xrightarrow[\delta^n]{} & \mathsf{H}^{n+1}(A') \end{array}$$

    commutes. (Note that despite the notation, δ^n depends also on the short exact sequence!)
  - Long exact sequence:

$$\cdots \longrightarrow \mathsf{H}^n(A) \longrightarrow \mathsf{H}^n(B) \longrightarrow \mathsf{H}^n(C) \xrightarrow{\delta^n} \mathsf{H}^{n+1}(A) \longrightarrow \cdots$$

- **Morphism between δ-functors** $\mathsf{H} \to \mathsf{H}'$: A family of functorial morphisms $\mathfrak{F}^n : \mathsf{H}^n \to \mathsf{H}'^n$ that commutes with δ i.e.

$$\begin{array}{ccc} \mathsf{H}^n(C) & \xrightarrow{\delta^n} & \mathsf{H}^{n+1}(A) \\ \downarrow{\scriptstyle \mathfrak{F}^n(C)} & & \downarrow{\scriptstyle \mathfrak{F}^{n+1}(A)} \\ \mathsf{H}'^n(C) & \xrightarrow{\delta'^n} & \mathsf{H}'^{n+1}(A) \end{array}$$

  for each short exact sequence.
- **Universal δ-functor**: H is universal if for every δ-functor H' and any morphism $\mathfrak{F}^0 : \mathsf{H}^0 \to \mathsf{H}'^0$ extend to a morphism $\mathsf{H} \to \mathsf{H}'$ of δ-functors.
- **Effacable functor**: Functor $\mathfrak{F} : \mathcal{A} \to \mathcal{B}$ such that for every $A \in \mathcal{A}$, there exists a monomorphism $u : A \to I$ such that $\mathfrak{F}(u) = 0$.
- **Left exact functor**: $\mathfrak{F}$ is left exact if for every exact sequence $0 \to A \to B \to C$ in $\mathcal{A}$, the sequence $0 \to \mathfrak{F}(A) \to \mathfrak{F}(B) \to \mathfrak{F}(C)$ is exact in $\mathcal{B}$.
- **Right derived functor** $(R^n\mathfrak{F})_{n\geq 0}$ of an additive functor $\mathfrak{F}$: The unique (if exists) universal δ-functor H with $\mathsf{H}^0 = \mathfrak{F}$.
- **Homological δ-functor, right exact functor, left derived functor**: Dual notions.

Theorem. *A δ-functor H is universal if H^n is effacable for every $n \geq 0$.*

Theorem. *Suppose that $\mathcal{A}$ has sufficiently many injectives i.e. for every $A \in \mathcal{A}$, there exists some injective object I and some mono $A \to I$. Then right derived functor $R^n\mathfrak{F}$ exists for every left exact functor $\mathfrak{F}$.*

For proof, take injective resolution and cohomology of the resulting chain complex. There's obviously dual theorem for left derived functor.

4. Spectral sequences

Let $\mathcal{A}$ be an abelian category.

- **Decreasing filtration** of an object $A \in \mathcal{A}$: A family $(A^n)_{n\in\mathbb{Z}}$ of subobjects of A such that $A^n \supseteq A^{n+1}$ for all n. For such a filtered object, denote

$$gr^n A := A^n/A^{n+1}.$$

- **Compatible morphism of filtered objects**: Morphism $f : A \to B$ such that $f(A_n) \subset B_n$.

- **Spectral sequence in $\mathcal{A}$**: A system
$$E = (E_r^{pq}, E_n, d_r^{pq}, \alpha_r^{pq}, \beta^{pq})$$
consisting of
  (1) Objects $E_r^{pq} \in \mathcal{A}$ for $p, q \in \mathbb{Z}$, $r \geq 2$;
  (2) Morphisms $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ such that $d \circ d = 0$ (these are called differentials);
  (3) Isomorphisms $\alpha_r^{pq} : \ker(d_r^{pq})/Im(d_r^{p-r,q+r-1}) \to E_{r+1}^{pq}$;
  (4) Filtered objects $E^n \in \mathcal{A}$, $n \in \mathbb{Z}$;
  (5) Isomorphisms $\beta^{pq} : E_\infty^{pq} \to gr_p E^{p+q}$ where $E_\infty^{pq} = E_r^{pq}$ for r large enough such that E_r^{pq} becomes constant independent of r, assuming $d_r^{pq}, d^{p-r,q+r-1} = 0$ as r gets sufficiently large.
- **Morphism between spectral sequences**: A system $\phi_r^{pq} : E_r^{pq} \to E_r'^{pq}$ and $\phi^n : E^n \to E'^n$ compatible with filtrations and commutes with d, α, β.

For a spectral sequence, one denotes
$$E_2^{pq} \Rightarrow E^{p+q}$$
The E_2^{pq} is called **initial term** and E^{p+q} is called **limit term** of the spectral sequence.

Intuitively, property (3) identifies E_{r+1}^{pq} with homology of E_r^{pq} so spectral sequences are typically compared with a book with infinitely many pages and the next page computes cohomology of previous page.

Property (5) says that the "graded pieces" of the limit term E^{p+q} can be computed by E_∞^{pq} which can be computed by taking "successive homology" of the pages until the result is stable.

References