SOME ANALYSIS

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1. Basic measure theory

Recall a *σ-algebra* over a set X is a collection Σ of subsets of X that contain $\emptyset$, closed under complement and countable intersection and union. If X is a topological space, the *Borel algebra* of X is the smallest σ-algebra over X generated by open sets of X.

Let F be either $\mathbb{R}$ (or $\mathbb{C}$). The category of F-measure space is the category whose objects are triple $(X, \Sigma, \mathfrak{M})$ where X is a set, Σ is a σ-algebra over X and $\mathfrak{M}$ is the Σ-sheaf of F-measurable functions on X given by

$$\mathfrak{M}(U) = \{f : U \to F \mid f^{-1}(V) \in \Sigma \text{ for every open } V \subset F\}$$

for all $U \in \Sigma$. A morphism $(X, \Sigma, \mathfrak{M}) \to (X', \Sigma', \mathfrak{M}')$ in this category is just a map $f : X \to X'$ such that $\phi \circ f \in \mathfrak{M}$ for all $\phi \in \mathfrak{M}'$ and is called *Σ-measurable functions*.

An *F-measure* μ on $(X, \Sigma, \mathfrak{M})$ is a map $\mu : \Sigma \to F \cup \{\infty\}$ that satisfies intuitive notion of a measure; for example, if $U_n \in \Sigma$ are disjoint then $\mu(\bigcup U_n) = \sum \mu(U_n)$. See any textbook on analysis such as Folland's for the full definition. With the notion of a measure, we can define Lebesgue integral; for example, if $f \geq 0$ and $\mu \geq 0$ then

$$\int f d\mu = \sup \left\{ \int s d\mu | 0 \leq s \leq f \right\}$$

where s are simple functions i.e. linear combination $\sum a_k 1_{S_k}$ of indicator functions of measurable sets and $\int s d\mu := \sum a_k \mu(S_k)$.

2. Haar measure

Let G be locally compact group so G is a natural $\mathbb{R}$-measure space via the Borel algebra. Then there exists a unique (up to scalar multiple) positive Radon[1] measure μ on G that is left invariant i.e. $\mu(gH) = \mu(H)$ for all Borel measurable H. Such measure is called the *left Haar measure*. By symmetry, there exists a right Haar measure ν which can be defined by $\nu(H) := \mu(H^{-1}) = \mu(\{h^{-1} \mid h \in H\})$.

Remark. I can't seem to get used to the notation of Getz as it does not fit into the framework of the usual definition of a measure. He wrote $d_\ell g$ for left Haar measure and defined $d_r g = d_\ell g^{-1}$. His notation should mean what I wrote above, by comparison with common literature.

In general, if $F(g)$ is a function $G \to G$ then Getz's notation $d_\ell F$ denotes the measure λ where $\lambda(H) = d_\ell(\{F(h) \mid h \in H\})$ where d_ℓ should be the left Haar measure.

From now on, all groups are implicitly assumed to be locally compact group.

Example 1. If G is a finite group equipped with discrete topology then its Haar measure is the counting measure.

[1]Finite on compact subset; outer regular $\mu(H) = \inf_{U \subset G \text{ open}} \mu(U)$ and inner regular $\mu(U) = \sup_{K \subset U \text{ compact}} \mu(K)$.

Example 2. The standard Haar measure of $(\mathbb{R}, +)$ is the Lebesgue measure. (Exercise: Prove this from the construction of Lebesgue measure.) The standard Haar measure of $(\mathbb{R}^\times, \times)$ is given by $\mu = \frac{dx}{|x|}$. Note that this means

$$\mu(U) = \int_U \frac{dx}{|x|}$$

for every Borel measurable subset $U \subset \mathbb{R}^\times$. The basic idea is of course to notice that the Lebesgue measure scale the measure of the scaled set aU by the factor $|a|$ so we divide to get a scale-invariant measure:

$$\mu(aU) = \int_{aU} \frac{dx}{|x|} = \int_U \frac{d(x/a)}{|x/a|} = \int_U \frac{dx}{|x|} = \mu(U).$$

Remark. (i) In the example above, dx refers to two different (but closedly related) things: a differential form on $\mathbb{R}$ and the Lebesgue measure on $\mathbb{R}$.

If we restrict to the multiplicative group $(\mathbb{R}^+, \times)$ then the form $\frac{dx}{|x|} = d\ln(x)$ is exact so by my interpretation of Getz's notation above, $\frac{dx}{|x|}$ should denote the measure which assigns to every measurable subset $U \subseteq \mathbb{R}^+$ the Lebesgue measure of the set $\{\ln(x) | x \in U\}$. (I was a bit confused at first that the measure might be negative; but this should not be the case. Also, the trick to get from $\mathbb{R}^+$ to the whole of $\mathbb{R}^\times$ is to use multiplication by -1.)

(ii) Generalizing the phenomenon above, it is easy to see that

Proposition 1. *If $\phi : G \to H$ is an isomorphism of topological group and μ is the left Haar measure on H then*

$$\nu(U) = \mu(\phi(U))$$

is the left Haar measure on G.

Proof. Obvious by transport of structure principle

$$\begin{aligned}\nu(gU) &= \mu(\phi(gU)) \\ &= \mu(\{\phi(gu) | u \in U\}) \\ &= \mu(\{\phi(g)\phi(u) | u \in U\}) \\ &= \mu(\phi(g)\phi(U)) \\ &= \mu(\phi(U))\end{aligned}$$

Note that here I use the assumption ϕ being an isomorphism to conclude that the set $\phi(U)$ is Borel measurable whenever U is; which also works if $G \to H$ is an *open embedding* but not when $\dim G \neq \dim H$. (Exercise: Think of how to get Haar measure for $\mathbb{R}$ from the Haar measure for $\mathbb{R}^2$.) □

Example 3. The standard Haar measure of $(\mathbb{C}, +)$ is (twice) the Lebesgue measure. Write $z = x + iy$ to identify $\mathbb{C} \cong \mathbb{R}^2$ then $2(dx \wedge dy)$ is Haar measure on $\mathbb{C}$. This is evident since each dx and dy are Haar measure on $\mathbb{R}$. As in the real case, with the same reasoning[2], the standard Haar measure for $(\mathbb{C}^\times, \times)$ is

$$\frac{2(dx \wedge dy)}{|z|^2} = \frac{2(dx \wedge dy)}{x^2 + y^2}.$$

(Note here that I am using the Euclidean distance for $|z|$. In number theory, people also define complex norm to be $|z|_{\mathbb{C}} = x^2 + y^2$ in which case the formula should read $\frac{2(dx \wedge dy)}{|z|}$.)

[2]But note that if we scale an area $U \subset \mathbb{C}^\times$ by z, the resulting Lebesgue area will be scaled by $|z|^2$.

Example 4. If F is non-archimedean local field of characteristic 0 (i.e. finite extension of $\mathbb{Q}_p$) with ring of valuation $\mathfrak{o}$ and uniformizer π then the (normalized) Haar measure is given by

$$\mu(a + \pi^k \mathfrak{o}_F) = q^{-k}$$

for every basic open set. Here, $q = |\mathfrak{o}/\pi|$ is the number of elements of the residue field. In characteristic 0, the formal exponential

$$\exp : \pi^h \mathfrak{o} \to 1 + \pi^h \mathfrak{o} : \ln$$

given by

$$\exp(x) = \sum \frac{x^k}{k!}$$

is an isomorphism for h large enough[3], we see that the Haar measure on $1 + \pi^h \mathfrak{o} \subset F^\times$ is similarly given by "integrating $\frac{dx}{|x|}$" by remark above (except that there is no "integration path" here[4]). Here, $|x|$ is the usual non-archimedean metric of x induced by the valuation i.e. given by $|x| = q^{-v_F(x)}$.

As with the situation of $\mathbb{R}^+$ and $\mathbb{R}^\times$, for any other basic open set U, there is an appropriate translate $aU \subset 1 + \pi^h \mathfrak{o}$.

With the above examples and patterns, one easily derive the Haar measure for general Lie group G over $\mathbb{R}$. Observe that the exponential map $\exp : \mathsf{T}G \to G$ yields a local diffeomorphism[5] between a neighborhood of $0 \in \mathfrak{g} \cong \mathbb{R}^{\dim G}$ and a neighborhood $U \subset G$ of the identity element of G. Therefore, the Haar measure on U *should be* induced by the Lebesgue measure of $\mathbb{R}^{\dim G}$ under the inverse of exp i.e. a kind of Lie group logarithm.

Remark. This is actually more subtle than it seems. The proposition requires an isomorphism as topological groups. There is no guarantee that the neighborhood U is a Lie subgroup of G: If it does not then we cannot make use of the property that "Haar measure on U is unique" to claim that our measure is the Haar measure on G restricted to U. Also, if U is not a subgroup then uS might go out of U for a measurable $S \subset U$; so it doesn't even make sense to talk about "Haar measure on U". The complete argument is the following: Let $U \subset G$ be a neighborhood of identity diffeomorphic to the unit ball in $\mathfrak{g}$; and let μ_U be the induced Lebesgue measure. Now G can be covered by a countable number of translates of U so for any measurable $S \subset G$, we can cover $S \subset \bigcup_i g_i U$ and then use infinitary (as limit of the usual finite) inclusion-exclusion principle to define the measure of S:

$$\mu(S) = \sum \mu_U(g_i^{-1} S \cap U) - \sum \mu_U(g_i^{-1} S \cap g_j^{-1} S \cap U) + \cdots$$

We have to show that this is well-defined measure on G that extends μ_U and that it is left invariant; hence the Haar measure of G. This definition could be seen to work in cases such as G being compact.

Example 5. For $F = \mathbb{R}$ or $F = \mathbb{C}$, the Haar measure on $\mathsf{GL}_n(F)$ is given by

$$\frac{\bigwedge_{i,j} dx_{ij}}{|\det x_{ij}|}$$

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[3] To be precise, $h \geq \lfloor \frac{e(F/\mathbb{Q}_p)}{p-1} \rfloor + 1$ where $e(F/\mathbb{Q}_p) = v_F(p)$ denotes the ramification index of the extension $F/\mathbb{Q}_p$. For example, $p\mathbb{Z}_p \cong 1 + p\mathbb{Z}_p$ if p is odd prime; whereas $4\mathbb{Z}_2 \cong 1 + 4\mathbb{Z}_2$. See Borevich-Shafarevich, Number Theory.

[4] Normally, we integrate over an n-form over an n-manifold. Here, integration is in measure theoretic sense and dx refers to Haar measure of the additive group $(F, +)$. One, of course, should recognize here that the function $\frac{1}{|x|}$ is dx-measurable; which is obvious.

[5] Look up the Gauss' lemma.

by geometric intuition: Think of $\mathsf{GL}_n(F)$ as subspace of the vector space F^{n^2} of all matrices which has Haar measure $\bigwedge_{i,j} dx_{ij}$. A linear transformation $g \in \mathsf{GL}_n(F)$ should scale the size of "figure in F^{n^2}" by $|\det g|$.

According to [1], this is also true when F is a non-archimedean local field or when F is the adele ring of a global field where the idele norm should be the obvious definition.

For instance, if F is a local field, what is the measure of the compact open subset $\mathsf{GL}_n(\mathfrak{o}_F)$ i.e. how does one compute $\int_{\mathsf{GL}_n(\mathfrak{o}_F)} \frac{\omega}{|\det x_{ij}|}$?

Remark. I have seen "$\mathsf{GL}_n(\mathfrak{o}_F)$ is open compact subgroup of $\mathsf{GL}_n(F)$" many times by now but have never tried to prove it for myself. Here is the proof.

Recall that the topology of $\mathsf{GL}_n(F)$ is given by its embedding into F^{n^2+1} where the matrix $(x_{ij}) \mapsto (x_{ij}, \det(x_{ij}))$. Basic open sets of F^{n^2+1} are product $\prod U_{ij} \times V$ where each $U_{i,j}, V$ are basic open sets of F. Hence, $\mathsf{GL}_n(\mathfrak{o}_F) = \mathfrak{o}_F^{n^2} \times F^\times \cap \mathsf{GL}_n(F)$ is open.

For compactness, just observe that $\mathfrak{o}_F^{n^2}$ is compact subset of F^{n^2} by Tychonoff's theorem so if $\mathsf{GL}_n(\mathfrak{o}_F) \subset \bigcup U_\alpha$ is an open cover then we can extract a finite cover for the projection $\rho(\mathsf{GL}_n(\mathfrak{o}_F)) \subset \bigcup_i \rho(U_{\alpha_i})$ where ρ extracts the matrix part i.e. drops the determinant in F^{n^2+1}. Since the determinant is determined from the matrix, the U_{α_i}'s automatically cover $\mathsf{GL}_n(\mathfrak{o}_F)$ in F^{n^2+1}.

Example 6. Let G be an affine algebraic group of dimension n over a global field F.

We have a unique (up to scalar multiple) left-invariant differential form $\omega \in \bigwedge^n \mathfrak{g}$. (Recall in differential geometry that a left invariant differential form is determined by its effect on the Lie algebra i.e. $\omega_e \in \bigwedge^n \mathfrak{g} : \mathfrak{g}^n \to \mathbb{R}$ or $\mathbb{C}$; similar to what we did for left invariant vector fields.)

Now $\mathfrak{g}$ is an F-vector space and $\omega : \mathfrak{g}^n \to F$ is an alternating linear form. By appropriate tensor, we get $\omega_v : \mathfrak{g}_v^n \to F_v$ for every place v of F. Assuming you understand what $d|\omega_v|$ i.e. the measure obtained from the form ω_v is, we can define the Haar measure on $G(\mathbb{A}_F)$: Pick a set S containing all infinite places and $K^S = \prod K_v$ maximal compact subgroup of $G(\mathbb{A}_F^S)$. Normalize $d|\omega_v|$ so that $d|\omega_v|(K_v) = 1$ for all $v \notin S$. Then $\prod_{v\in S} d|\omega_v|$ is the Haar measure on $G(\mathbb{A}_F)$.

Getz seems to conflate the two notions of differential form and measure. Unfortunately, he also did not explain the meaning of the notation $\prod_{v\in S} d|\omega_v|$.

3. Modulus character

G is called *unimodular* if the left and right Haar measure coincide. In particular, if G is abelian, it is unimodular. In fact, any compact group is unimodular; for if μ is left invariant, the measure

$$\nu : U \mapsto \int_G \mu(Ug) d\mu$$

[i.e. integral of the function $f : G \to \mathbb{R}; g \mapsto \mu(Ug)$] is both left and right invariant; hence, $\nu = c\mu$ for some constant c. The constant c can be determined as the quotient

$$c = \frac{\nu(G)}{\mu(G)} = \frac{\int_G \mu(Gg)d\mu}{\mu(G)} = \frac{\int_G \mu(G)d\mu}{\mu(G)} = \frac{\mu(Gg)^2}{\mu(G)} = \mu(G)$$

and we see that the compactness requirement is apparent.

When G is not unimodular, we have a quasi-character $\delta_G : G \to \mathbb{R}^+$, that measure the discrepancy between left and right measure, defined in the following way: $\delta_G(h) \in \mathbb{R}^+$ is the number such that

$$d_\ell(Sh) = \delta_G(h)\; d_\ell(S)$$

for any measurable set S. Here, d_ℓ is the fixed left Haar measure of G. Such a number exists since for every fixed h, the induced measure $\mu_h : S \mapsto d_\ell(Sh)$ is left invariant and so must be a constant multiple of d_ℓ.

Remark. What I defined here might be the inverse of the modular character defined by Getz.

4. Convolution

Just to record the definition: Getz defined the convolution on $L^1(G, d_r)$ as

$$f_1 * f_2(g) := \int_G f_1(gh^{-1}) f_2(h) d_r h$$

where d_r is right Haar measure on G.

5. Ending remark

In practice, we almost never work with the explicit construction of Haar measure. Most likely, we pick a compact subgroup K (which must have finite measure) and normalize (i.e. declare) its measure to 1. Then for any other set, we try to relate it with the measure of K. For example, if $H \subset K$ is a finite index subgroup then

$$\mu(H) = \frac{1}{[K : H]} \mu(K).$$

More generally, if H is commeasurable with K i.e. $H \cap K$ has finite index in both H and K then

$$\mu(H) = [H : H \cap K] \mu(H \cap K) = \frac{[H : H \cap K]}{[K : H \cap K]} \mu(K).$$

This probably works well enough for the finite part. For the infinite part, one probably needs to do more elaborated integral computation.

It would be great to have a summary of how one performs these kinds of computation.

What is the relationship between Haar measure on quotient group G/H and Haar measure of G? For instance, what is the Haar measure on $\mathsf{GL}_n(\mathbb{A}_{\mathbb{Q}})/\mathsf{GL}_n(\mathbb{Q})$?

References

[1] Jayce R. Getz. An introduction to automorphic representations, 2011. Available at http://math.duke.edu/j̃getz/aut_reps.pdf.