# Automorphic Representations IV

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We will give a treatment of the theory so far in the special case $G = \mathsf{GL}_2$ over $\mathbb{Q}$. TODO: Add literature for about $G = \mathsf{GL}_2$ over $K$ or more general $\mathsf{GL}_n$.

## 1. Review

We recall the basic facts about $\mathsf{GL}_2$ over any field $F$.

(i) $G$ is reductive. Thus, we have an equivalence of automorphic representations as
- irreducible subquotient of $\mathcal{L}^2([G])$; and as
- irreducible admissible $(\mathfrak{g}, K) \times G(\mathbb{A}_F^\infty)$-modules that is a subquotient of $\mathcal{L}^2([G])$.

**Remark 1.1.** In [1] or Getz's course notes, one finds another definition: as an admissible representation of $\mathcal{H}$ isomorphic to a subquotient of $\mathcal{L}^2(G(F)A_G\backslash G(\mathbb{A}_F))$ where $\mathcal{H}$ is the *global Hecke algebra*[1].

(ii) The adelic quotient
$$[G] := G(F)\backslash G(\mathbb{A}_F)^1$$
where
$$G(\mathbb{A}_F)^1 := \bigcap_{\chi \in X^*(G)} \ker(|\cdot| \circ \chi : G(\mathbb{A}_F) \to \mathbb{G}_m(\mathbb{A}_F) \to \mathbb{R}_+)$$
is unimodular and
$$|\cdot| := \prod |\cdot|_v : \mathbb{G}_m(F)\backslash \mathbb{G}_m(\mathbb{A}_F) \to \mathbb{R}_+$$
is the adele norm homomorphism.

(iii) $A_G \leq G(F_\infty)$ is the identity component of the $\mathbb{R}$-points of the maximal $\mathbb{Q}$-split torus in $\mathsf{Res}_{F/\mathbb{Q}} Z_G$ where $Z_G$ is center of $G$. We have
$$A_G G(\mathbb{A}_F)^1 = G(\mathbb{A}_F)$$
if $G$ is reductive so in that case
$$A_G\backslash G(\mathbb{A}_F) \cong G(\mathbb{A}_F)^1$$
and
$$[G] \cong G(F)A_G\backslash G(\mathbb{A}_F)$$
has finite volume.

(iv) $T = \left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} | t_1, t_2 \in \mathbb{G}_m \right\}$ is a maximal torus of $G$. So $G$ is split.

(v) The subgroup $B$ of upper triangular matrices is a Borel subgroup of $G$. Iwasawa decomposition $G(\mathbb{R}) = B(\mathbb{R})\mathsf{O}(\mathbb{R})$, $G(\mathbb{C}) = B(\mathbb{C})\mathsf{U}(\mathbb{C})$ and $G(L) = B(K)G(\mathcal{O}_L)$ if $L$ is a non-archimedean local field.

[1] In the number field case, $\mathcal{H} = \mathcal{H}_\infty \otimes \mathcal{H}^\infty$ where $\mathcal{H}_\infty := H(G(\mathbb{R} \otimes_Q F), K_\infty)$ is the Hecke algebra at infinity consisting of distributions of the Lie group $G(\mathbb{R} \otimes_Q F)$ supported on $K_\infty$ and $\mathcal{H}^\infty := \mathcal{C}_c^\infty(G)$ is the non-archimedean Hecke algebra consisting of locally constant compactly supported functions.

(vi) The Lie algebra of $G$ is $\mathfrak{g} = M_2(F)$.

If $\mathfrak{g}$ is a Lie algebra over a field $F$ then the universal enveloping algebra

$$U(\mathfrak{g}) = T(\mathfrak{g})/I$$

where $T(\mathfrak{g}) = F \oplus \mathfrak{g} \oplus \mathfrak{g}^{\otimes 2} \oplus \cdots$ is the (graded) tensor algebra, $\mathfrak{g}^{\otimes n} = \mathfrak{g} \otimes \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ for $n$ times and $I$ is the ideal generated by $X \otimes Y - Y \otimes X - [X,Y]$ for every $X, Y \in \mathfrak{g}$.

Note that we have canonical map $\mathfrak{g} \to U(\mathfrak{g})$ and it satisfies the universal property that any morphism of Lie algebras $\varphi : \mathfrak{g} \to A$ i.e. such that $\varphi([X,Y]) = \varphi(X)\varphi(Y) - \varphi(Y)\varphi(X)$ factors through $\mathfrak{g} \to U(\mathfrak{g})$.

**Warning**: $U(\mathfrak{gl}_n) \neq \mathfrak{gl}_n$.

(vii) Strong approximation: If $F$ is a number field then

$$\mathsf{GL}_2(F) \backslash \mathsf{GL}_2(\mathbb{A}_F)/K$$

is finite for any compact open $K = K_\infty K^\infty$. Let $t_1, ..., t_h$ be a complete set of representatives. Then we have homeomorphism

$$\bigsqcup_{i=1}^{h} \Gamma_i(K^\infty) A_G \backslash G(F_\infty) \to G(F) A_G \backslash G(\mathbb{A}_F)/K^\infty$$

$$\Gamma_i(K^\infty) A_G \, g_\infty \mapsto [g_\infty t_i]$$

where $\Gamma_i(K^\infty) = G(F) \cap t_i A_G \backslash G(F_\infty) K_\infty t_i^{-1}$ is an arithmetic subgroup.

## 2. Representation theory of $\mathsf{GL}_2(\mathbb{R})$

2.1. **Maximal compact subgroup.** Recall that $K = \mathsf{O}_2(\mathbb{R})$ is maximal compact subgroup of $\mathsf{GL}_2(\mathbb{R})$. Note that $K$ is the subgroup generated by rotations $\mathsf{SO}_2(\mathbb{R})$ and the reflection $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. It follows from our previous description of unitary irreducible representations of $\mathsf{SO}_2(\mathbb{R})$ that all but one unitary irreducible representations of $K$ are two-dimensional: For each $\ell \in \mathbb{Z}$, put

$$V_\ell := \begin{cases} \mathbb{C}v_0 & \text{if } \ell = 0 \\ \mathbb{C}v_\ell \oplus \mathbb{C}v_{-\ell} & \text{if } \ell \neq 0 \end{cases}$$

then we have the unitary irreducible representation

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto e^{i\ell\theta} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \begin{cases} v_\ell \mapsto v_{-\ell} \\ v_{-\ell} \mapsto v_\ell \end{cases}.$$

2.2. **Admissible $(\mathfrak{g}, K)$-modules.** Recall that $\mathfrak{g}$ is just the vector space of matrices $M_2(\mathbb{R})$. We take

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for its basis. Set $\Delta := \frac{1}{4}(H^2 + 2XY + 2YX)$ then $Z(\mathfrak{g}) = \langle \Delta, Z \rangle$. For each pair of complex numbers $s, \mu \in \mathbb{C}$, let

$$\chi_{s,\mu} : Z(\mathfrak{g}) \to \mathbb{C}$$

be the character such that $Z \mapsto \mu$ and $\Delta \mapsto s(1-s)$. For each $\epsilon \in \{0,1\}$, we have the $(\mathfrak{g}, K)$-module with infiniestimal character $\chi_{s,\mu}$ where

$$V_{s,\mu,\epsilon} = \bigoplus_{\ell \equiv \epsilon \bmod 2} \mathbb{C} v_\ell$$

and

$$Xv_\ell = (s + \frac{\ell}{2})v_{\ell+2} \qquad Yv_\ell = (s - \frac{\ell}{2})v_{\ell-2}$$
$$\Delta v_\ell = s(1-s)v_\ell \qquad Zv_\ell = \mu v_\ell$$

$V_{s,\mu,\epsilon}$ is irreducible unless $s = k/2$, $k \equiv \epsilon \bmod 2$.

In case $V_{s,\mu,\epsilon}$ is reducible, it has a unique irreducible infinite dimensional sub-representation $V_k$ and the quotient $V_{s,\mu,\epsilon}/V_k$ is finite dimensional.

**Definition 2.1.** If $V_{s,\mu,\epsilon}$ is irreducible, it is called *principal series representation*. Otherwise, the irreducible sub-representation $V_k$ is called *discrete series representation* (if $k \neq 1$) or *limit of discrete series* (if $k = 1$).

All (infinite dimensional) $(\mathfrak{g}, K)$-modules are either principal series, discrete series or limit of discrete series. (Note that there is a trivial 1-dimensional $(\mathfrak{g}, K)$-modules.)

2.3. **Note.** The above computation is derived in [1] by studying admissible representation[2] of Hecke algebra at infinity $\mathcal{H}(\mathsf{GL}_{2/\mathbb{R}}) = U(\mathfrak{g}^{\mathbb{C}}) \oplus \epsilon * U(\mathfrak{g}^{\mathbb{C}})$ where $\mathfrak{g}^{\mathbb{C}}$ is complexification of $\mathfrak{g}$ and $\epsilon$ is the Dirac measure at $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ to take care of two components of $\mathsf{GL}_2(\mathbb{R})$. It follows from Harish-Chandra's Subquotient Theorem that all irreducible admissible representations of $\mathsf{GL}_2(\mathbb{R})$ are subquotient of "principal series" $H(\mu_1, \mu_2) = \mathfrak{Ind}_B^G(\mu_1\mu_2)$ where $\mu_1\mu_2$ is the representation of Borel subgroup $B$ given by $\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \mapsto \mu_1(t_1)\mu_2(t_2)$.

More precisely, let $\mu_1, \mu_2$ be (continuous) quasi-characters of $\mathbb{R}^\times$. It follows that $\mu_i(t) = |t|^{s_i}\mathrm{sgn}(t)^{m_i}$ for some $s_i \in \mathbb{C}$ and $m_i \in \{0,1\}$. Set $\mu = \mu_1\mu_2^{-1}$ and $s = s_1 - s_2, m = m_1 - m_2$. The representation space

$$H(\mu_1, \mu_2) = \left\{ K\text{-finite } \varphi : G \to \mathbb{C} \;\middle|\; \varphi(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} g) = \mu_1(t_1)\mu_2(t_2) \left|\frac{t_1}{t_2}\right|^{1/2} \varphi(g) \right\}$$

and the action of $X \in \mathfrak{g}$ is the natural $(X\varphi)(g) = \frac{d}{dt}\big|_{t=0} \varphi(g \exp(-tX))$. From Iwasawa decomposition $B = NA$ with $A$ diagonal and $G = NA\mathsf{SO}_2$, the function in $H(\mu_1, \mu_2)$ only depends on its effect on $\mathsf{SO}_2$. The functions $\{\varphi_n | n \equiv m \bmod 2\}$ where $\varphi_n(R_\theta) = e^{-in\theta}$ gives a basis for $H(\mu_1, \mu_2)$. Then theorem 4.4 and 4.5 of [1] says that

(i) if $\mu$ is not of the form $t \mapsto t^p\mathrm{sgn}(t)$ with $p \neq 0$ then $H(\mu_1, \mu_2)$ is irreducible;
(ii) if $\mu$ is not of the form $t \mapsto t^p\mathrm{sgn}(t)$ and $p > 0$ then $H(\mu_1, \mu_2)$ has exactly 1 invariant subspace spanned by $H^s = \{\varphi_{-p-3}, \varphi_{-p-1}, \varphi_{p+1}, ...\}$ and the quotient is denoted by $\pi(\mu_1, \mu_2)$;
(iii) if $\mu$ is not of the form $t \mapsto t^p\mathrm{sgn}(t)$ and $p < 0$ then the only invariant subspace is $H^f = \{\varphi_{p+1}, \varphi_{p+3}, ..., \varphi_{-p-3}, \varphi_{-p-1}\}$.

[2] How to go back?

The principal series consists of the irreducible representation in case (i), the finite dimensional quotient representation on $H/H^s$ in case (ii), and the finite dimensional sub-representation in case (iii). These representations are denoted $\pi(\mu_1, \mu_2)$.

The discrete series is the infinite dimensional sub-representation on $H^s$ in case (ii) or the quotient $H/H^f$ in case (iii). These representations are denoted $\sigma(\mu_1, \mu_2)$.

Every irreducible admissible representation is either $\pi(\mu_1, \mu_2)$ or $\sigma(\mu_1, \mu_2)$. The relation between them are

$$\pi(\mu_1, \mu_2) \cong \pi(\mu_2, \mu_1)$$

and

$$\sigma(\mu_1, \mu_2) \cong \sigma(\mu_2, \mu_1) \cong \sigma(\mu_1 \cdot \mathrm{sgn}, \mu_2 \cdot \mathrm{sgn}) \cong \sigma(\mu_2 \cdot \mathrm{sgn}, \mu_1 \cdot \mathrm{sgn}).$$

$\pi(\mu_1, \mu_2)$ corresponds to unitary representation of $G$ if and only if both $\mu_1, \mu_2$ are unitary or $t = s_1 + s_2$ imaginary and $s = s_1 - s_2$ is real, non-zero and is in $[-1, 1]$.

$\sigma(\mu_1, \mu_2)$ corresponds to unitary representation of $G$ if and only if the central character is unitary i.e. $t = s_1 + s_2$.

## 3. Representation theory of $\mathsf{GL}_2(\mathbb{C})$

In this case, $\mathcal{H}(G) = U(\mathfrak{g}^{\mathbb{C}})$ since $\mathsf{GL}_2(\mathbb{C})$ is connected.

We similarly have "principal series" $\rho(\mu_1, \mu_2)$ which is irreducible unless $\mu(z) = z^p \overline{z}^q$ with $p, q \in \mathbb{Z}$, $pq > 0$. And every admissible representations is subquotient of some $\rho(\mu_1, \mu_2)$.

In case it is reducible, there is one irreducible sub-quotient which is actually equivalent to some $\rho(\mu_1', \mu_2')$. Thus, there is no discrete series.

## 4. Non-archimedean representation theory of $\mathsf{GL}_2$

Let $G = \mathsf{GL}_2(F)$ where $F$ is a non-archimedean local field. The Hecke algebra $\mathcal{H}(G)$ in this case is just the space $\mathcal{C}_c^\infty(G)$ of locally constant compactly supported functions. (Recall that admissible representation of $G$ corresponds one-to-one to admissible representation of $\mathcal{H}(G)$ where $\pi \mapsto (f \mapsto \int_G f(g)\pi(g)d^*g)$ where $d^*g$ is the Haar measure assigning 1 to $\mathsf{GL}_2(\mathcal{O}_F)$.)

Just like the archimedean case, for $\mu_1, \mu_2$ quasi-characters of $F^\times$, we get a representation $\rho(\mu_1, \mu_2) = \mathfrak{Ind}_B^G \mu_1\mu_2$ on the space

$$H(\mu_1, \mu_2) = \{\text{locally constant } \varphi : G \to \mathbb{C} | \varphi() = ...\}$$

Then theorem 4.18 of [1] says that $\rho(\mu_1, \mu_2)$ is irreducible (in which case it is called *principal series*) unless $\mu = |x|$ or $\mu = |x|^{-1}$.

(i) If $\mu = |x|^{-1}$ then $\rho$ contains 1-dimensional invariant subspace and the quotient is irreducible.
(ii) If $\mu = |x|$ then $\rho$ contains irreducible invariant subspace of co-dimension 1.

The irreducible representations in (i) and (ii) are called *special representations.*

Every irreducible admissible representation of $G$ are either principal series, special representations above or *supercuspidal representation* i.e. for every $v$, there exists open compact subgroup $U \subset N$ such that $\int_U \pi(n)vdn = 0$ (equivalently, the matrix coefficients $(\pi(g)v, \widetilde{v})$ are compactly supported (modulo the center). The terminology implies that supercuspidal representations are not equivalent to subquotient of any $\rho(\mu_1, \mu_2)$.

## 5. Automorphic Representation of $\mathsf{GL}_2$

Now let $F$ be a global field and let $\pi_v$ irreducible unitary (hence admissible) representation of $\mathsf{GL}_2(F_v)$ on some $H_v$ such that $\pi_v$ is *class 1* i.e. *spherical* for almost every $v$ i.e. the representation $\pi_v|_K$ contains the identity at most once.

**Remark 5.1.** An infinite dimensional irreducible admissible representation of $G$ (over non-archimedean local field) is class 1 if and only if $\pi = \pi(\mu_1, \mu_2)$ for some *unramified* characters $\mu_1, \mu_2$ and $\pi$ is not a special representation.

Then we can form the representation $\pi = \bigotimes' \pi_v$ on $H = \widehat{\bigotimes' H_v}$ (Hilbert space completion). This is an admissible representation of $\mathsf{GL}_2(\mathbb{A}_F)$ since $H(\sigma) = \bigotimes H_v(\sigma)$ is finite dimensional for all $\sigma = \bigotimes \sigma_v$. (Also recall that every irreducible admissible representation of $\mathsf{GL}_2(\mathbb{A}_F)$ is factorizable with $\pi_v$ uniquely determined.) Thus, we have a mean to construct all automorphic representations using local representations.

Of course, it is much important to know what kind of information an automorphic representation give us and to construct automorphic representation having arithmetic significance.

We now study the first definition of automorphic representations as irreducible subquotient of the regular representation $R$ of $G(\mathbb{A})$ on

$$\mathcal{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = \int_{\substack{\psi:\mathbb{Q}^\times\backslash Z_G\to\mathbb{C} \\ \text{central character}}} \mathcal{L}(G(\mathbb{Q})\backslash G(\mathbb{A}), \psi)\, d\psi.$$

Let $R^\psi$ be the regular representation on each $\mathcal{L}(G(\mathbb{Q})\backslash G(\mathbb{A}), \psi)$ and $R_0^\psi$ the restriction on cuspidal subspace.

**Fact.** $R_0^\psi$ is discrete direct sum of irreducible unitary representations each with finite multiplicity. This is more or less a reinterpretation of the basic fact that the spaces of cusp forms are finite dimensional.

**Fact.** Irreducible constituents of $R_0^\psi$ are admissible (by our equivalence of definitions); in particular, factorizable.

**Theorem 5.2** (Multiplicity One, Theorem 5.7 in [1])**.** *The multiplicity of each irreducible constituents of $R^\psi$ is one.*

**Remark 5.3.** The proof of the above theorem given in [1] makes use of the uniqueness of *Whittaker model*. Also, the theorem is not valid for other groups such as $\mathsf{GSp}_4$, in which case, I heard from conferences that people study *Bessel model*.

## 6. Automorphic Forms of $\mathsf{GL}_2$

As an application of Theorem 5.2, one obtains a one-to-one correspondence between

- new-forms in $S_k(N, \psi)$ where $\psi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$ is a Dirichlet character mod $N$; and
- irreducible constituent of $R_0^{\widehat{\psi}}$ where $\widehat{\psi} = \prod \psi_p$ is a certain central grossencharacter corresponding to $\psi$.

### 6.1. From elliptic modular forms to automorphic forms and to representation.

Observe that $A_{\mathsf{GL}_{2/\mathbb{Q}}} = \mathbb{R}_+ I$ is the group of positive scalar matrices. We get a homeomorphism

$$A_{\mathsf{GL}_{2/\mathbb{Q}}} \backslash \mathsf{GL}_2(\mathbb{R}) / \mathsf{O}_2(\mathbb{R}) \cong \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

and the upper half plane allows us to view a modular form in $f \in M_k(\Gamma)$ as a function $f : A_{\mathsf{GL}_{2/\mathbb{Q}}} \backslash \mathsf{GL}_2(\mathbb{R}) \to \mathbb{C}$. Define $\varphi_f : \mathsf{GL}_2^+(\mathbb{R}) \to \mathbb{C}$ be the function defined by

$$\varphi_f(g) = j(g, i)^{-k} f(gi)$$

for all $g \in \mathsf{GL}_2^+(\mathbb{R})$ where if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $j(g, z) := \sqrt{\det(g)}(cz + d)$ is the familiar factor of automorphy and the action $gz = \frac{az+b}{cz+d}$.

Then $\varphi_f \in \mathcal{A}_\infty(\Gamma, \langle \Delta - \frac{1}{4}(k^2 - 1), Z\rangle, \sigma_k)$ is a (classical) automorphic form. TODO: Modify Getz's presentation to account for the character.

To get the adelic automorphic form, suppose that $f \in S_k(N, \psi)$ is a new-form. Let $K = K_0(N)$. In this case, $\mathsf{GL}_2(\mathbb{Q}) \backslash \mathsf{GL}_2(\mathbb{A}) / K$ is a singleton by strong approximation; in other words, every $g \in \mathsf{GL}_2(\mathbb{A})$ can be written as $g = \gamma k_0$ where $\gamma \in \mathsf{GL}_2(\mathbb{Q})$ and $k_0 \in K$. Define $\varphi_f : G(\mathbb{Q}) \backslash G(\mathbb{A}) \to \mathbb{C}$ by the formula

$$\varphi_f(g) := j(\gamma, i)^{-k}\, f(\gamma i)\, \widehat{\psi}(k_0).$$

TODO: Define $\widehat{\psi}$. Then $\varphi_f \in \mathcal{L}^2_{\text{cusp}}(G(\mathbb{Q}) \backslash G(\mathbb{A}), \widehat{\psi})$. Let $H(f)$ be subspace spanned by $G(\mathbb{A})\varphi_f$ and $\pi_f$ the representation on $H(f)$. Then $\pi_f$ is an irreducible constituent of $R_0^\varphi$ ($\pi_f$ has the same local components at $\infty$ and $v \nmid N$). (See Theorem 5.19 of [1].)

**Remark 6.1.** We know that $\pi_f = \otimes_v' \pi_v$ is factorizable. It is not easy to determine the local representation $\pi_v$ in general. But in case $N$ is square-free then

- $\pi_\infty$ is discrete series representation with $p = k - 1$ and $t = 0$;
- $\pi_p = \pi(\mu_1, \mu_2)$ is the spherical representation $\mu_1\mu_2$ is trivial if $(p, N) = 1$; and
- $\pi_p$ is the special representation with trivial central character if $p|N$.

## 7. The future

*Is undecided.*

Thank you for having fun studying automorphic representations this semester.

## References

[1] Gelbart, S. S. *Automorphic forms on adèle groups*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1975. Annals of Mathematics Studies, No. 83.