Automorphic Representations III

LAWRENCE VU

1. Representation theory of td groups

When G is an algebraic group over non-archimedean local field F then $G(F)$ is totally disconnected. In fact, it satisfies

Definition 1.1. A group G is called a *td group* if every neighborhood of the identity contains a compact open subgroup.

Remark 1.2. According to Getz, this is equivalent to being Haussdorff + locally compact + totally connected. As a result, $G(\mathbb{A}_F^\infty)$ is also a td group.

General td group does not have extra structure like Lie group (smoothness) so we can't talk about derivative (or transport it to the representation space naturally). It turns out that the analogy of smooth functions in this context are locally constant functions. Unlike Getz, my policy is to use distinct term so I will prefix "td" to all the redefined concept e.g. "td-admissible" is used instead of "admissible".

Definition 1.3. A representation is *td-smooth* if the stabilizer of every vector is open i.e.

$$V = \bigcup_K V^K$$

Definition 1.4. A representation of a td group G is *td-admissible* if it is td-smooth and the space of K-fixed vector V^K is finite dimensional for every compact open subgroup K of G.

1.1. Hecke algebra.

We have seen that representation of G leads to $\mathcal{C}_c(G)$-module in natural way. If we restrict to the smooth subalgebra $\mathcal{C}_c^\infty(G)$ of $\mathcal{C}_c(G)$ then we have an equivalent of category with smooth representations.

Definition 1.5. A function $f \in \mathcal{C}_c(G)$ is *td-smooth* if it is locally constant.

We use the notation $\mathcal{C}_c^\infty(G)$ to denote the space of such smooth functions. As before, the space $\mathcal{C}_c^\infty(G)$ can be made into an $\mathbb{C}$-algebra via convolution as product. It is called the *Hecke algebra* of G. If K is a compact open subgroup of G, we denote $\mathcal{C}_c^\infty(G//K) \subset \mathcal{C}_c^\infty(G)$ the sub-algebra of functions that are left and right invariant by K. It is easy to see that

$$\mathcal{C}_c^\infty(G) = \bigcup_{K \text{ compact open}} \mathcal{C}_c^\infty(G//K).$$

Remark 1.6. Note that both $\mathcal{C}_c(G)$ and $\mathcal{C}_c^\infty(G)$ do not have unit i.e. a function $e : G \to \mathbb{C}$ such that the convolution $e * f = f$ for all f. On the contrary, the function $e_K := \frac{1}{\mu(K)} 1_K$ is the identity on $\mathcal{C}_c^\infty(G//K) = e_K \mathcal{C}_c^\infty(G) e_K$. Here, $1_K : G \to \mathbb{C}$ denotes the characteristic function of K. Neither $\mathcal{C}_c^\infty(G)$ nor $\mathcal{C}_c^\infty(G//K)$ is guaranteed to be commutative!

The full Hecke algebra is hard (its structure is not easy to describe in general) so its module are evidently hard so one normally study modules over the subalgebra $\mathcal{C}_c^\infty(G//K)$ instead.

Fact. A smooth G-module V is irreducible if and only if V^K is irreducible $\mathcal{C}_c^\infty(G//K)$-module for all compact open subgroup $K \leq G$.

1.2. **Flath's Theorem.** We shall now describe the statement of Flath's factorization theorem in more details. First, the restricted tensor product of infinitely many modules or algebras can be defined as the inverse limit.

Theorem 1.7 (Flath's factorization theorem)**.** *Every admissible irreducible representation π of $\mathcal{C}_c^\infty(G(\mathbb{A}_F^\infty))$ is factorizable i.e. $\pi = \bigotimes_v' \pi_v$.*

The key idea is that

$$\mathcal{C}_c^\infty(G(\mathbb{A}_F^\infty)) \cong \bigotimes_v{}' \mathcal{C}_c^\infty(G(F_v))$$

2. Second Definition of Automorphic Representations

The following definition assumes G to be *reductive* over a number field F. Let $K_\infty \leq G(F_\infty)$ be maximal compact subgroup and $\mathfrak{g} = \operatorname{Lie} G_{F_\infty}$. Note that $G(F_\infty) = \prod_{v|\infty} G(F_v) = G(\mathbb{R})^{r_1} G(\mathbb{C})^{r_2}$ so $K_\infty = \prod K_v$ where each K_v is maximal compact subgroup of $G(F_v)$. TODO: What should be the definition for general G?

A $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$-module is naturally a vector space V with actions of $(\mathfrak{g}, K_\infty)$ and $G(\mathbb{A}_F^\infty)$ that commute with each other. It is admissible if V^{K^∞} is admissible $(\mathfrak{g}, K_\infty)$-module for every compact open $K^\infty \leq G(\mathbb{A}_F^\infty)$.

Definition 2.1. An automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ is an irreducible admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ module isomorphic to a subquotient of $\mathcal{L}^2([G])$.

To pass from the original definition to the new definition, we take the (adelic) subspace of K-finite vector but we need to explain this in the context of $G(\mathbb{A}_F)$ (we previously defined it for Lie group): Let $\pi : G(\mathbb{A}_F) \to \mathsf{GL}(V)$ be a representation, $K_\infty \leq G(F_\infty)$ and $K^\infty \leq G(\mathbb{A}_F^\infty)$ be maximal compact subgroups and $K := K_\infty K^\infty$. A vector $v \in V$ is called K-finite if the subspace $\langle \pi(k)v \mid k \in K \rangle$ of its K-translate is finite dimensional.

Remark 2.2. Reductive assumption is used here to show that the two definitions are equivalent via $V \to V_{\text{fin}}$. In particular, it is to apply Harish-Chandra's theorem that all irreducible unitary representation of $G(F)$ is admissible (if F is non-archimedean and G is reductive). TODO: My guess is that to change the notion of admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$-module appropriately when G is not reductive, namely requiring that it is td-admissible.

3. Automorphic Forms

We shall describe an example of automorphic representation using Definition 2.1 via the space of automorphic forms. There are two notions of automorphic forms: Let $U(\mathfrak{g})$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ and $Z(\mathfrak{g})$ be its center.

- **Classical**: Let $\Gamma \subset G(F)$ be arithmetic subgroup. An *classical automorphic form* on [of level] Γ is a smooth function $\varphi : G(F_\infty) \to \mathbb{C}$ that is

(i) of moderate growth i.e. there are constants $c, r \in \mathbb{R}_+$ such that $|\varphi(g)| \leq c\|g\|^r$ where $\|\cdot\|$ is a certain metric on $G(F_\infty)$ defined via a linear representation $G \to \mathsf{GL}_{2n}$;
(ii) left Γ invariant i.e. $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \Gamma$;
(iii) K_∞-finite i.e. $\langle K_\infty \varphi \rangle \cong \sigma$ for some $\sigma \in \widehat{K_\infty}$; and
(iv) $Z(\mathfrak{g})$-finite i.e. the space $\langle Z(\mathfrak{g})\varphi \rangle$ is finite dimensional; equivalently, φ is annihilated by some ideal J of $Z(\mathfrak{g})$ of finite codimension (note that elements of $\mathfrak{g}$ are differential operators and since φ is smooth, $Z(\mathfrak{g})\varphi$ will be functions).

We denote $\mathcal{A}_\infty(\Gamma, J)$ the space of classical automorphic forms of level Γ and annihilated by the ideal J. It decomposes

$$\mathcal{A}_\infty(\Gamma, J) = \bigoplus_{\sigma \in \widehat{K_\infty}} \mathcal{A}_\infty(\Gamma, J, \sigma)$$

- **Adelic**: An *automorphic form* is a smooth function $A_G \backslash G(\mathbb{A}_F) \to \mathbb{C}$ that is
  (i) of moderate growth (analogous definition, the only thing that changes is the metric on $A_G \backslash G(\mathbb{A}_F)$);
  (ii) left $G(F)$ invariant;
  (iii) K-finite where $K = K_\infty K^\infty$ (note that this means the space spanned by the functions $\{x \mapsto \varphi(xk) | k \in K\}$ is finite dimensional; and
  (iv) $Z(\mathfrak{g})$-finite (if F is a number field).

  We denote $\mathcal{A}(J)$ the space of automorphic forms annihilated by the ideal J.

We have the following result:

Theorem 3.1. *Let $J \leq Z(\mathfrak{g})$ be an ideal of finite codimension. Then*
(i) $\mathcal{A}_\infty(\Gamma, J)$ *is an admissible* $(\mathfrak{g}, K_\infty)$*-module.*
(ii) $\mathcal{A}(J)$ *is an admissible* $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$*-module.*

Remark 3.2. Note that we didn't say $\mathcal{A}(J)$ gives an automorphic representation here! Not all automorphic forms are square integrable i.e. in $\mathcal{L}^2([G])$, so we can't identify $\mathcal{A}(J)$ with a sub-quotient of $\mathcal{L}^2([G])$, as in the definition. The subspace of *cusp forms* $\mathcal{A}_{\text{cusp}}(J)$ is dense in $\mathcal{L}^2_{\text{cusp}}([G])$ and will give us cuspidal representations.

References