Automorphic Representations II

LAWRENCE VU

One principle in mathematics is to work in the category that takes into account "the full structure of objects". For example, in group cohomology, we work with continuous co-chain should the group be a topological group. In the same manner, for topological groups, we want to study the subcategory of "continuous representations" whose objects are continuous group homomorphism $G \to \mathsf{GL}_{\text{cont}}(V)$ and for Lie groups, the (wishful) subcategory of "smooth representations" within the category of "continuous representations". Here, $\mathsf{GL}_{\text{cont}}(V) \subset \mathsf{GL}(V)$ is the subspace of continuous invertible linear maps $V \to V$, which has natural topology[1] i.e. $\mathsf{GL}_{\text{cont}}(V)$ is a topolgical space so it makes sense to talk about continuous maps. As a convention, we always assume representation to be continuous in this note since all groups concerned are topological group.

The problem with Lie group case is that it is hard to find the right subspace $\mathsf{GL}(V)$ that has "differential" structure i.e. a closed subgroup $W \subset \mathsf{GL}(V)$ that is also a smooth manifold as in differential geometry. Worst, the representation space we are dealing with is $V = \mathcal{L}^2(X, \mu)$ for certain measure space X which is supposed to be infinite dimensional[2] so there is no hope to find such W which by definition needs to be finite dimensional $\mathbb{R}$-manifold, *unless one generalizes the notion of differentiability.* Note that V itself doesn't even have differential structure so $\mathsf{GL}_{\text{smooth}}(V)$ doesn't make sense.

1. The case of compact Lie group

When G is compact[3] Lie group, the representation theory (in the category of continuous representations, we don't even need extra-structure) is quite simple. Observe that

Lemma 1.1. *If G is compact then the representation π is unitarizable. More generally, if K is any compact subgroup of G then we can adjust the inner product on V so that the restriction representation $\pi|_K$ is unitary representation of K.*

To see that, let $[,]$ be the original paring on the V and we define a new paring by averaging $\langle v, w\rangle := \int_K [\pi(k)v, \pi(k)w] \, dk$. We state the following major result

Theorem 1.2 (Peter-Weyl theorem)**.** *Suppose that G is compact Lie group. Then*

- (i) *Any irreducible unitary representation of G is finite dimensional.*
- (ii) *The matrix coefficients (i.e. functions $G \to \mathbb{C}$ of the form $\langle \pi(g)v, w\rangle$ for some fixed v, w) of finite dimensional unitary representations of G are dense in $\mathcal{C}(G)$ and $\mathcal{L}^p(G)$ for all $1 \le p \le \infty$.*
- (iii) *If $\pi : G \to \mathsf{GL}(V)$ is an unitary representation then V decomposes into a Hilbert space direct sum of irreducible unitary sub-representations.*

[1] As long as V is nice, I assume. The topology is called "strong operator topology" by Terry Tao in his blog `https://terrytao.wordpress.com/tag/peter-weyl-theorem/`.

[2] Why?

[3] Note that "compact" is the continuous analogue of being "finite". But I am not saying that representation theory of finite group is simple here.

Remark 1.3. The statement above was copied from Getz. I did a bit research on that and I found that one proof of statement (i) doesn't seem to require Lie group structure `https://mathoverflow.net/questions/119402/why-all-irreducible-representations-of-compact` only on Schur's lemma. In other words, it applies to any compact topological group. According to Wikipedia, the whole Peter-Weyl theorem applies to all compact topological group, not only compact Lie groups, as written in Getz. (If you can read German, please check the original article by Peter and Weyl [**?**] to verify this claim.)

Remark 1.4. The statement of Peter-Weyl theorem is about *continuous* representations. The emphasis here is it applies to all continuous representations, not only "smooth" or "admissible" (to be defined) ones. I don't think the result is true if we consider all discontinuous representations but one should not care about those kind of representations by the principle of mathematics.

Remark 1.5. Explicit computations in practice seem to make heavy use of the representation of the Lie algebra (see below).

TODO: Unfortunately, I don't know how to go from representation of Lie algebra to representation of the group. According to Wikipedia, even for finite dimensional representations, this correspondence is not one-to-one, unless G is simply connected: There are Lie algebra representations that do not come from Lie group representation.

Example 1.6. Let us consider the case $G = \mathsf{SO}_2 = \{g \in \mathsf{GL}_2 \mid gg^t = I_2, \det g = 1\}$ over $\mathbb{R}$; explicitly

$$\mathsf{SO}_2(\mathbb{R}) = \left\{ R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \ \theta \in [0, 2\pi] \right\}$$

is isomorphic to a circle S^1 and so is *abelian*. We know that irreducible representation of abelian groups are characters. In this case, we find that irreducible unitary representations of $\mathsf{SO}_2(\mathbb{R})$ are of the form $R_\theta \mapsto e^{ik\theta}$ (i.e. scalar multiplication) where $k \in \mathbb{Z}$.

2. Representation theory of Lie groups

Let G be affine algebraic group over archimedean local field F (for now, just think of $G = \mathsf{GL}_n$ and $F = \mathbb{R}$) and $\pi : G(F) \to \mathsf{GL}(V)$ be a representation of the Lie group $G(F)$ on a Hilbert space V. As usual, denote by $\mathfrak{g}$ for the Lie algebra of $G(F)$. Subsequently, we shall use G to refer to $G(F)$ whenever a group is expected. In this section, we describe the passage from the (admissible) representation π to an (admissible) $(\mathfrak{g}, K)$-module where K is any maximal compact subgroup of G.

2.1. Admissible Representations.

Using Peter-Weyl theorem, one obtains a key idea to study representation of G that is to see its restriction to maximal compact subgroup of G. Let K be any maximal compact subgroup of G.

If π is a representation of G then without loss of generality (by lemma), one can assume $\pi|_K$ is unitary so by Peter-Weyl theorem, it splits

$$\pi|_K = \bigoplus \sigma$$

where each σ are irreducible unitary (hence, finite dimensional) representation of K. Note that the σ are allowed to repeat and the multiplicities need not be finite.

Remark 2.1. An important question is what kind of "decomposition" one can have for a general irreducible representation π?

We define the "σ-eigenspace"

$$V(\sigma) := \{v \in V \mid \langle \pi(K) \cdot v \rangle \cong \sigma\}$$

Here, the $\langle \pi(k) \cdot v \rangle$ is the subspace spanned by $\pi(k) \cdot v$ for all $k \in K$ and the isomorphism are as representations of K. The vectors of $V(\sigma)$ are said to have *K-type* σ and K-finite. Note that the space of $V(\sigma)$ only depends on the restriction $\pi|_K$.

Definition 2.2. The representation π is *admissible* if $V(\sigma)$ is *finite dimensional* for all $\sigma \in \widehat{K}$.

Here, $\widehat{K}$ is the *unitary dual* of K, the set of equivalence classes of irreducible (unitary) representations of K. For instance, we found in ?THM? **??** that $\widehat{\mathsf{SO}_2(\mathbb{R})} \cong \mathbb{Z}$.

Fact. There is a well-known theorem of Harish-Chandra that all unitary representation are admissible for reductive group G.

Remark 2.3. The definition of admissible representation does NOT depend on the choice of the maximal compact subgroup K even though the definition above makes a choice of K; otherwise, I should have used the term "K-admissible".

2.2. $(\mathfrak{g}, K)$**-module.** Now we define the second concept of

Definition 2.4. A $(\mathfrak{g}, K)$-module V is a Hilbert space V with action of $\mathfrak{g}$ and K

$$\pi : \mathfrak{g} \to \mathsf{End}(V) \qquad \rho : K \to \mathsf{GL}(V)$$

such that

(i) $V = \bigoplus_i V_i$ is a countable direct sum with each V_i finite dimensional, K-invariant;
(ii) $\pi(X) \cdot v = \frac{d}{dt}\big|_{t=0} \rho(\exp(tX)) \cdot v$ for all $X \in \mathrm{Lie}\,(K)$; and
(iii) $\rho(k)\pi(X)\rho(k) = \pi(\mathrm{Ad}(k)X)$.

A $(\mathfrak{g}, K)$-module is called *admissible* if we can choose V_i to have distinct K-types.

Remark 2.5. As we noted earlier, the notion of K-type only depend on the K action on V i.e. $V_i = V(\sigma)$ only depends on the representation ρ of K in the above definition. Also, it is evident that a $(\mathfrak{g}, K)$-module depends on the choice of K.

2.3. **From admissible representation to admissible** $(\mathfrak{g}, K)$**-module.** Let V_{fin}, *the subspace of K-finite vectors*, be the algebraic direct sum

$$V_{\text{fin}} := \bigoplus_{\sigma \in \widehat{K}} V(\sigma).$$

Then V_{fin} is an admissible $(\mathfrak{g}, K)$-module. (See Proposition 4.4.1 in [**?**].)

Fact. For the converse direction, an admissible $(\mathfrak{g}, K)$-module can be canonically identified with the space of K-finite vectors of certain admissible representation of G on a smooth Frechet space of moderate growth (theorem of Casselman and Wallach [**?**]).

Remark 2.6. Two admissible representations having the same $(\mathfrak{g}, K)$-module are called *infinitestimally equivalent*. In other words, there could be in-equivalent admissible representations leading to isomorphic $(\mathfrak{g}, K)$-module. But this can only happen when G is not reductive by Harish-Chandra [**?**]: If G is reductive then infinitestimally equivalent representations are unitarily equivalent.

References

[1] Peter, F., and Weyl, H. Die Vollständigkeit der primitiven Darstellungen einer geschlossenen kontinuierlichen Gruppe. *Math. Ann. 97*, 1 (1927), 737–755.