AUTOMORPHIC FORMS AND REPRESENTATIONS I

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Having all the basic objects (algebraic groups, Haar measure, etc.), we are ready to define the central notion in the theory: automorphic representation of an algebraic group G.
Notation: If (X, μ) is a measure space, we use the standard notation for the Lebesgue space

$$\mathcal{L}^r(X, \mu) := \left\{ f : X \to \mathbb{C} \text{ measurable} \;\middle|\; \int_X |f(x)|^r d\mu < \infty \right\}$$

for all $r \in \mathbb{R}^+$. When G is a locally compact Hausdorff group, the right Haar measure is implicitly understood $\mathcal{L}^r(G) = \mathcal{L}^r(G, d_r g)$.

1. Group Representation

We recall various notion related to group representation.

Definition 1. Let V be a vector space (not necessarily finite dimensional, mostly concern with $\mathbb{C}$-vector space) and G be a group. A *representation* π of G on V is a group homomorphism $\pi : G \to \mathsf{GL}(V)$.

If V is equipped with a bi-linear pairing $(,)$ (say, a Hilbert space) then a representation π is called *unitary* if $(\pi(g)v, \pi(g)w) = (v, w)$. Equivalently, it means that $\pi : G \to \mathsf{U}(V)$ where $\mathsf{U}(V)$ is the subgroup of $\mathsf{GL}(V)$ consisting of unitary operators.

Definition 2. If π_1, π_2 are representations of G on V_1, V_2 respectively (over the same field) then a *morphism* $\pi_1 \to \pi_2$ is a linear map $V_1 \to V_2$ such that the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow{I} & V_2 \\ \downarrow{\scriptstyle \pi_1(g)} & & \downarrow{\scriptstyle \pi_2(g)} \\ V_1 & \xrightarrow{I} & V_2 \end{array}$$

commutes for every $g \in G$. Such a map I is called an *intertwining map*.

With these definitions, for each group G, we have the category $\mathbf{Repns}_G$ of representations of G. When G has more structure (for example, G is a topological group or Lie group), we have natural subcategory (e.g. continuous representations or smooth representations).

Definition 3. Let $\pi : G \to \mathsf{GL}(V)$ be a representation. A representation $\rho : G \to \mathsf{GL}(W)$ is called

- a *sub-representation* of π if W is a subspace of V and $\rho(g) = \pi(g)|_W$ for all $g \in G$;
- a *quotient representation* if there is a surjection $V \to W$; and
- a *sub-quotient* if it is a sub-representation of a quotient representation.

Note that a sub-representation (quotient representation, resp.) defined above are categorical sub-object[1] (quotient object, resp.) in the category of representations $\mathbf{Repns}_G$.

[1]To be sub-object, we need an injective intertwining map $I : W \to V$ and so W is subspace of V.

Example 1. Let G be locally compact Hausdorff group and X is a right G-space equipped with a G-invariant measure μ.

Then we have the following representation

$$\pi : G \to \mathsf{GL}(\mathcal{L}^2(X,\mu)) \qquad \pi(g)f := (x \mapsto f(x \cdot g))$$

It is *continuous*[2] and is called the *regular representation.* (In the case of G being a finite group, a regular representation of G is the familiar notion of a *group action.*)

Equipped $\mathcal{L}^2(X,\mu)$ with the standard inner product

$$(f,g) = \int_X f(x)\overline{g(x)}d\mu$$

then π is unitary.

Example 2. One principal technique to get new representation is by *induction* from a representation of a subgroup. (By a result of Langlands, we get all "good representations" by parabolic induction.)

Observe that if $\pi : G \to \mathsf{GL}(V)$ is a representation of G then we have a representation $\pi|_H : H \to \mathsf{GL}(V)$ of any subgroup H by restriction. In other words, we have a functor

$$\mathfrak{Res} : \mathbf{Repns}_G \to \mathbf{Repns}_H$$

The adjoint operation[3] to restriction is *induction* i.e. the functor

$$\mathfrak{Ind} : \mathbf{Repns}_H \to \mathbf{Repns}_G$$

such that

- $\mathfrak{Ind}$ is right adjoint to $\mathfrak{Res}$ i.e.

$$\mathsf{Hom}(\mathfrak{Res}(\pi), \pi') = \mathsf{Hom}(\pi, \mathfrak{Ind}(\pi')) \tag{1}$$

  if π and π' are representations of G and H respectively (equality here means "in bijection with")
- $\mathfrak{Ind}$ is left adjoint to $\mathfrak{Res}$ i.e.

$$\mathsf{Hom}(\pi, \mathfrak{Res}(\pi')) = \mathsf{Hom}(\mathfrak{Ind}(\pi), \pi') \tag{2}$$

  if π and π' are representations of H and G respectively.

We shall construct the functor $\mathfrak{Ind}$ explicitly below. The fact that it is adjoint functor (i.e. (1) and (2) hold) is known as *Frobenius Reciprocity.* (The proof for finite group G normally goes through theory of $K[G]$-modules and is quite complicated.)

If $H \subseteq G$ is a (closed) subgroup and $\pi : H \to \mathsf{GL}(V)$ is a representation of H then we define the representation

$$\mathfrak{Ind}_H^G \pi : G \to \mathsf{GL}(W)$$

where

$$W = \{\phi : G \to V | \phi \in \mathcal{L}^2(G/H) \text{ and } \phi(gh^{-1}) = \pi(h)\phi(g) \forall g \in G, h \in H\}$$

and the action is the obvious one

$$(\mathfrak{Ind}_H^G \pi)(g)\ \phi : x \mapsto \phi(g^{-1}x)$$

for any $\phi \in W$.

[2] [1], Proposition 2.41.

[3] In category theory, a functor $\mathfrak{F} : \mathcal{C} \to \mathcal{D}$ is called left adjoint to a functor $\mathfrak{G} : \mathcal{D} \to \mathcal{C}$ if there is a bijection between $\mathsf{Hom}_{\mathcal{D}}(\mathfrak{F}A, B)$ and $\mathsf{Hom}_{\mathcal{C}}(A, \mathfrak{G}B)$ for any objects $A \in \mathcal{C}$ and $B \in \mathcal{D}$.

2. Representation of $\mathcal{C}_c(G)$ induced from representation of G

All groups are assumed to be locally compact Haussdorff. Let $\mathcal{L}^1(G)$ be the space of $\mathbb{C}$-valued measurable function on G with finite measure and $\mathcal{C}_c(G)$ be the subspace of compactly supported functions. Note that $\mathcal{C}_c(G)$ is an associative $\mathbb{C}$-algebra with product being convolution.

A representation π of G on V induced a $\mathbb{C}$-algebra homomorphism $\psi : \mathcal{C}_c(G) \to \mathsf{End}(V)$ given by

$$\psi(f) : v \mapsto \int_G \pi(g) f(g) v \, d\mu$$

where μ is the right Haar measure on G.

Remark. This construction is the analogy of the classical construction in representation theory of finite groups where given a representation $G \to \mathsf{GL}(V)$ where V is a vector space over the field K, we can then view V as a natural $K[G]$-module where

$$K[G] = \left\{ \sum_{i=1}^{n} a_i g_i | a_i \in K, g_i \in G \right\}$$

is the group algebra. Conversely, a $K[G]$-module V provides a natural representation $G \to \mathsf{GL}(V)$.

The idea is that for finite group, a function $f \in \mathcal{C}_c(G)$ is just a finite collection of complex number $(f(g))_{g\in G}$ so it can be identified with a finite formal sum

$$f = \sum_{g \in G} f(g) \, g \in \mathbb{C}[G].$$

Recall that Haar measure for finite group is just counting measure. So we see that

$$\begin{aligned} \pi(f)(v) &= \int_G \pi(g) f(g) v \, d\mu \\ &= \left(\sum_{g \in G} f(g) g \right) v \end{aligned}$$

is the familiar $\mathbb{C}[G]$-algebra structure on V alluded above.

Under this realization, one can see how classical results of representation theory for finite groups generalize to locally compact Hausdorff topological groups e.g. Frobenius Reciprocity. We shall come back to this when we talk about non-archimedean representation theory.

3. First Definition of Automorphic Representation

Now we give the first definition of automorphic representation. Let G be affine algebraic group over a global field F and define the adelic quotient

$$[G] := G(F) \backslash G(\mathbb{A}_F)^1$$

Then we have Hilbert space $\mathcal{L}^2([G])$ and a regular representation of $G(\mathbb{A}_F)^1$ on it via Example 1.

Definition 4. An *automorphic representation* of $G(\mathbb{A}_F)^1$ is an irreducible unitary representation of $G(\mathbb{A}_F)^1$ that is equivalent to a sub-quotient of (the regular representation of $G(\mathbb{A}_F)^1$ on) $\mathcal{L}^2([G])$.

At this point, it is hard to write down an explicit example[4] of an automorphic representation and it is beyond me to tell you its importance; except by quoting Langlands Conjecture about a correspondence between "automorphic forms" and "Galois representations".

[4]Is the regular representation irreducible?

To achieve that, we shall need some preparation as well as an alternative (more "algebraic") definition via $(\mathfrak{g}, K) \times G(\mathbb{A}_F^\infty)$-module.

I want to highlight a major problem here is that one needs to have representation theory for groups with extra-structures (real-complex Lie groups, algebraic group, reductive groups over non-archimedean local fields, ...) which is different from the classical representation theory of finite groups. The first thing one does is to find the "right" subcategory of $\mathbf{Repns}_G$ to work with. (The category of all representations is very hard to study and does not take into account important arithmetical aspects of the groups we are interested in.) Such subcategory arises from Harish-Chandra and Jacquet-Langlands' work: the category of *admissible representations*.

Our next goal is to explain

Theorem 1 (Flath's factorization theorem)**.** *Every admissible irreducible representation* π *of* $\mathcal{C}_c^\infty(G(\mathbb{A}_F^\infty))$ *is factorizable i.e.* $\pi = \bigotimes_v' \pi_v$.

Using Theorem 1, we only need to construct appropriate local representations π_v which together with an $(\mathfrak{g}, K)$-module structure describe an automorphic representation.

As a remark, there is another systematic method to get automorphic representations via *automorphic forms* and hence, via modular forms.

References

[1] Gerald B. Folland. *A course in abstract harmonic analysis.* Textbooks in Mathematics. CRC Press, Boca Raton, FL, second edition, 2016.