# ALGEBRAIC GROUP V - ADELIC POINTS AND STRONG APPROXIMATION

LAWRENCE VU

The [hopefully finally] last phase of algebraic groups is to discuss adelic points and approximation.

## 1. Adele

Suppose that $K$ is a global field. We define the adele ring of $K$ as the topological ring

$$\mathbb{A}_K := \prod_{v|\infty} K_v \times \prod_{v\nmid\infty}' K_v$$

where the restricted direct product is with respect to the valuation ring $\mathfrak{o}_{K,v}$ of $K_v$. Alternatively, let

$$\widehat{\mathbb{Z}} := \lim_{\substack{\leftarrow \\ n}} \mathbb{Z}/n\mathbb{Z} \cong \prod_p \mathbb{Z}_p \qquad \text{and} \qquad \mathbb{A}_{\mathbb{Z}} := \widehat{\mathbb{Z}} \times \mathbb{R}$$

then

$$\mathbb{A}_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \qquad \text{and} \qquad \mathbb{A}_K = \mathbb{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} K.$$

In fact, $\mathbb{A}_F = \mathbb{A}_K \otimes_K F$ for any finite extension $F/K$; and all equalities here indicate isomorphisms as topological ring.

If $S$ is a finite set of place, we write

$$\mathbb{A}_K^S := \prod_{v\notin S}' K_v \qquad \text{and} \qquad K_S := \prod_{v\in S} K_v.$$

**Fact.**
- (i) $\mathbb{A}_K$ is locally compact Hausdorff.
- (ii) The image of the diagonal embedding $K \to \mathbb{A}_K$ is discrete.
- (iii) The image of $K \to K_S$ is dense. By Chinese Remainder Theorem.
- (iv) The image of $K \to \mathbb{A}_K^S$ is dense for any $S \neq \emptyset$.

The last two statements will be known as weak and strong approximation for the additive group $\mathbb{G}_a$.

## 2. Adelic Points

If we have an algebraic group $G$ defined over $K$ or its ring of integers $\mathfrak{o}_K$, then we can study its group of adelic points $G(\mathbb{A}_K)$ since $\mathbb{A}_K$ is natural $K$ or $\mathfrak{o}_K$ algebra. At this point, this is just a group so the first step is to turn it into a topological group.

There is a canonical way to do it (for any affine scheme) by Theorem 2.2.1 in [2]. The idea is to take embedding the affine scheme $X(R) \leftrightarrow \mathrm{Hom}(A, R) \to R^A$ and take the subspace topology. The uniqueness in Theorem 2.2.1 means that the topology we get does not depend on the embedding of $X$.

**Example 1.** Let us consider $X = \mathsf{GL}_n = \mathrm{Spec}(A)$ where $A = K[x_{ij}, y]/(\det(x_{ij})y - 1)$ is the familiar ring. If $R$ is a topological ring, then we can embed $\mathsf{GL}_n(R) \to R^{n^2+1}$. The canonical topology for $\mathsf{GL}_n(R)$ is the subspace topology.

In particular, the topology of $\mathsf{GL}_1(\mathbb{A}_K) = \mathbb{A}_K^\times$ is the subspace topology of the embedding $\mathbb{A}_K^\times \to \mathbb{A}_K^2; x \mapsto (x, x^{-1})$. It turns out that this topology is the same as the restricted product $\prod_{v|\infty} K_v^\times \times \prod'_{v\nmid\infty} K_v^\times$ with respect to $\mathfrak{o}_{K,v}^\times$; and that this is NOT the same as subspace topology of $\mathbb{A}_K^\times \to \mathbb{A}_K$. This situation is not unique to $\mathsf{GL}_1$ i.e. the fact that the canonical topology on $G(\mathbb{A}_K)$ can be expressed as a restricted direct product. We have the more general fact below.

**Fact.** For any algebraic group $G$, let $G \to \mathsf{GL}_n$ be a faithful representation and let $H_v$ be the intersection of $\mathsf{GL}_n(\mathfrak{o}_{K,v})$ and (the image of) $G(K_v)$ in $\mathsf{GL}_n(K_v)$. Then

$$G(\mathbb{A}_K) \cong \prod_v{}' G(K_v)$$

with respect to $H_v \subset G(K_v)$.

## 3. Approximation in Algebraic Groups

**Definition 1.** An affine scheme $X$ is said to satisfied

(i) *weak approximation* with respect to $S$ if the image of $X(F) \to X(F_S)$ is dense; and
(ii) *strong approximation* if $X(F) \to X(\mathbb{A}_F^S)$ is dense.

Strong approximation is equivalent to saying that for any compact open subgroup $U$ of $G(\mathbb{A}_F^S)$, we have [c.f. Shimura's or Exercise 2.12]

$$G(\mathbb{A}_F^S) = G(F)U.$$

**Example 2.** According to Shimura, the groups $\mathsf{SL}_n$, $\mathsf{Sp}_n$ and $\mathsf{SU}_n$ are known to have strong approximation. In particular, from strong approximation for $\mathsf{SL}_n$, we have the familiar "strong approximation for $\mathsf{GL}_2$": If $U \leq \mathsf{GL}_2(\mathbb{A})$ is such that $\det U \supset \widehat{\mathbb{Z}}^\times$ then

$$\mathsf{GL}_2(\mathbb{A}) = \mathsf{GL}_2(\mathbb{Q})\ \mathsf{GL}_2(\mathbb{R})\ U.$$

More generally, let $U \subset \mathsf{GL}_n(\mathbb{A}_K^\infty)$ be compact open subgroup such that $\det(U) \supset \widehat{\mathfrak{o}_K}^\times = \prod_{v\nmid\infty} \mathfrak{o}_{K,v}^\times$, then

$$\mathsf{GL}_n(K)\mathsf{GL}_n(K_\infty)\backslash\mathsf{GL}_n(\mathbb{A}_K)/U = h_K = |\mathrm{Cl}_K|$$

according to Bump's [1], Theorem 3.3.1.

## 4. Iwasawa decomposition

If $G$ is a reductive group over a global field $K$ and $P \leq G$ a parabolic subgroup. Then there exists a maximal compact subgroup $U \leq G(\mathbb{A}_K)$ such that

$$G(\mathbb{A}_K) = P(\mathbb{A}_K)U.$$

This is known as Iwasawa decomposition.

**Example 3.** Let $B \leq \mathsf{GL}_n$ denote the Borel subgroup of upper triangular matrices. Then for a local field $L$ with valuation ring $R$ (if applicable),

$$\mathsf{GL}_n(L) = B_n(L) \begin{cases} O_n(\mathbb{R}) & \text{if } L = \mathbb{R}, \\ U_n(\mathbb{R}) & \text{if } L = \mathbb{C}, \\ \mathsf{GL}_n(R) & \text{if } L \text{ is non-archimedean.} \end{cases}$$

where $O_n = \{g \in \mathsf{GL}_n | gg^t = I_n\}$ and $U_n = \{g \in \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathsf{GL}_n | g\bar{g}^t = I_n\}$ are the standard groups of orthogonal and unitary linear transformations. From here, one easily deduces Iwasawa decomposition for $\mathsf{GL}_n$ over a global field $K$ and the parabolic subgroup $B_n$, namely

$$U = \prod_{v \text{ real}} O_n(\mathbb{R}) \times \prod_{v \text{ complex}} U_n(\mathbb{C}) \times \prod_{v<\infty} \mathsf{GL}_n(\mathfrak{o}_{K,v}).$$

## 5. Arithmetic Subgroups

Let $G \leq \mathsf{GL}_n$ be linear algebraic group. A subgroup $\Gamma \leq G(K)$ is called *arithmetic* if it is commeasurable with $\mathcal{G}(\mathfrak{o}_K)$ where $\mathcal{G}$ denotes the schematic closure of $G$ in $\mathsf{GL}_{n/\mathfrak{o}_K}$.

I don't want to explain schematic closure. But congruence subgroups are examples of arithmetic groups.

## References

[1] D. Bump. *Automorphic Forms and Representations*. Automorphic Forms and Representations. Cambridge University Press, 1998.

[2] Jayce R. Getz. An introduction to automorphic representations, 2011. Available at http://math.duke.edu/j̃getz/aut_reps.pdf.