ALGEBRAIC GROUPS IV - ROOT DATUM AND CLASSIFICATION

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1. Tori

Suppose G is algebraic group over k. We denote the set of characters

$$X^*(G) = \mathrm{Hom}_{\mathbf{AlgGrp}_k}(G, \mathbb{G}_m)$$

and co-characters

$$X_*(G) = \mathrm{Hom}_{\mathbf{AlgGrp}_k}(\mathbb{G}_m, G).$$

A torus of rank n is an algebraic group T such that $T_{k^{\mathrm{sep}}} \cong \mathbb{G}_m^n$; equivalently, $T_L \cong \mathbb{G}_m^n$ for some finite extension L/k. T is called

(i) *split* if $T \cong \mathbb{G}_m^n$ (or equivalently, $X^*(T)_k \cong \mathbb{Z}^{rank(T)}$, see remark);
(ii) *anisotropic* if $X^*(T)_k = \{\mathrm{id}\}$;
(iii) *maximal* in G if $T_{\overline{k}}$ is maximal amongst tori of $G_{\overline{k}}$.

Remark. There is an equivalence of categories

$$\{\mathrm{Tori}/k\} \leftrightarrow \{\text{finite rank } \mathbb{Z}\text{-torsion free } \mathbb{Z}[\mathrm{Gal}(k^{\mathrm{sep}}/k)]\text{-modules}\}$$

given by $T \mapsto X^*(T)_{k^{\mathrm{sep}}}$.

Remark. In case it is not obvious that $X^*(T) \cong \mathbb{Z}^{rank(T)}$, let me illustrate that

$$\mathrm{Hom}_k(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}.$$

Recall that $\mathbb{G}_m$ is represented by $\mathrm{Spec}A$ where $A = k[x,y]/(xy-1) \cong k[x,x^{-1}]$ can be viewed as the ring of "polynomials in x and x^{-1}". A morphism of algebraic groups $\mathbb{G}_m \to \mathbb{G}_m$ must be at least a scheme homomorphism which is equivalent to a k-algebra homomorphism $A \to A$ and I claim that

$$\mathrm{Hom}_k(A, A) \cong \mathbb{Z} \qquad [x \mapsto x^m] \mapsto m$$

Evidently, a ring homomorphism $k[x,x^{-1}] \to k[x,x^{-1}]$ is entirely determined by the image of x which must be in $k[x,x^{-1}]^\times$. Now I claim that $f \in k[x,x^{-1}]$ is invertible if and only if $f = ax^n$ for some $a \in k^\times$ and $n \in \mathbb{Z}$. Write $f(x) = \sum_{i=m}^{n} a_i x^i$ where $m,n \in \mathbb{Z}, m < n, a_m \neq 0$; and suppose that $g(x) = \sum_{i=m'}^{n'} b_j x^j$ [with similar restrictions] is its inverse i.e. $fg = 1$. Then the term of minimal degree in fg is $x^{m+m'}$ with coefficient $a_m b_{m'} \neq 0$; so we can only have $fg = 1$ when $m + m' = 0$ and $a_m b_{m'} = 1$ and $f = a_m x^m$. To be an k-algebra homomorphism, we can only send $x \mapsto x^m$. It is easy to check that all these morphisms give rise to morphisms of group schemes $\mathbb{G}_m \to \mathbb{G}_m$.

Fact. Every connected algebraic group admits a maximal torus and all maximal tori in $G_{\overline{k}}$ are conjugate under $G(\overline{k})$.

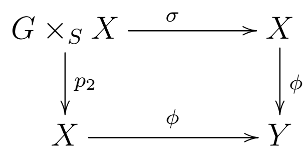
2. Weyl group

Suppose that G is connected reductive group and $T \leq G$ is a maximal torus. We define the *Weyl group* of T in G to be

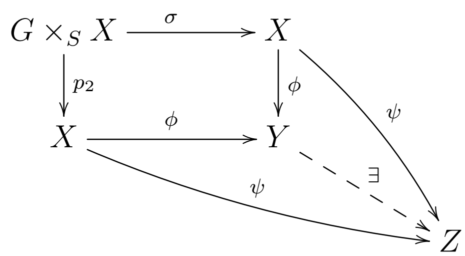
$$W(G,T) := N_G(T)/Z_G(T)$$

where we recall that $N_G(T)$ and $Z_G(T)$ are normalizer and centralizer of T in G which are algebraic groups (Chapter VII, Section 6 of [1]).

The quotient here is *scheme-theoretic quotient* which I sort of skipped when talking about group schemes: Suppose that G is a group scheme over some base scheme S, acting on a scheme X over S. A pair (Y, ϕ) where Y/S and S-morphism $X \to Y$ is called a *categorical quotient* of X by G if



commutes and is universal to that diagram i.e. if there is another pair $(Z, \psi : X \to Z)$ that can replace (Y, ϕ) in the above diagram then it factors through Y, diagrammatically



Intuitively, Y is "X with G-orbits identified": If we think of $G \times_S X$ as "set of tuples (g, x)" then the diagram says that the image of $\sigma(g, x) = g \cdot x$ and $p_2(g, x) = x$ in Y are the same. There are other notions such as *geometric quotient*, *universal categorical quotient*, ...; see [2]. Wikipedia article `https://en.wikipedia.org/wiki/Geometric_invariant_theory` provides some good explanation on the subject which I shall include in the appendix.

Chapter VIII of Milne [1] shows the existence of quotient of G by a normal subgroup N (under assumptions about k being a field, G smooth, etc.). For the case at hand, the "group $Z_G(T)$" acts on the scheme $N_G(T)$, hence on its ring of regular functions $\mathcal{O}(N_G(T))$, and as a scheme, $W(G,T)$ is just $\mathrm{Spec}\mathcal{O}(N_G(T))^{Z_G(T)}$, where $\mathcal{O}(N_G(T))^{Z_G(T)}$ are elements of $\mathcal{O}(N_G(T))$ "fixed by $Z_G(T)$". (See the appendix for intuition.)

Fact. It turns out that $W(G,T)$ is a smooth group scheme over k. But it is hard to describe the group $W(G,T)(R)$ for all k-algebra R; but take these for granted:

(i) $W(G,T)(k^{\mathrm{sep}}) = N_G(T)(k^{\mathrm{sep}})/Z_G(T)(k^{\mathrm{sep}})$, and
(ii) $W(G,T)(L) = W(G,T)(k^{\mathrm{sep}})^{\mathrm{Gal}(k^{\mathrm{sep}}/L)}$ for any intermediate field extension $k \subset L \subset k^{\mathrm{sep}}$.

3. Root datum

3.1. **Root system.** Let V be a finite dimensional $\mathbb{R}$-vector space and $\Phi \subset V$.

Definition 1. The triple $(\Phi, V, \{s_\alpha : V \to V\}_{\alpha \in \Phi})$ is called a *root system* if

(i) Φ if finite, $0 \notin \Phi$, $Span(\Phi) = V$;
(ii) For any $\alpha \in \Phi$, $s_\alpha(\alpha) = -\alpha$, the subspace $\{x \in V \mid s_\alpha(x) = x\}$ is 1-dimensional and $s_\alpha(\Phi) = \Phi$;
(iii) For any $\alpha, \beta \in \Phi$, one has $s_\alpha(\beta) - \beta = n\alpha$ for some $n \in \mathbb{Z}$.

Definition 2. The *Weyl group* of a root system (Φ, V, s_α) is the subgroup of $\mathsf{GL}(V)$ generated by reflections
$$W(\Phi, V) := \langle s_\alpha | \alpha \in \Phi \rangle.$$

3.2. **Root datum.** Let X, Y be free abelian groups with a perfect pairing $\langle , \rangle : X \times Y \to \mathbb{Z}$ and $\Phi, \Phi^\vee$ are sets with a bijection
$$\Phi \to \Phi^\vee \qquad \alpha \mapsto \alpha^\vee.$$

For any $\alpha \in \Phi$, we define $s_\alpha : X \to X$ and $s_{\alpha^\vee} : Y \to Y$ by
$$s_\alpha(x) := x - \langle x, \alpha^\vee \rangle \alpha$$
$$s_{\alpha^\vee}(y) := y - \langle \alpha, y \rangle \alpha^\vee$$

(The book has a mistake in the order of α and y in the definition of $s_{\alpha^\vee}$.)

Definition 3. A quadruple $(X, Y, \Phi, \Phi^\vee)$ is a *root datum* if

(i) $\langle \alpha, \alpha^\vee \rangle = 2$;
(ii) for each $\alpha \in \Phi$, $s_\alpha(\Phi) \subset \Phi$ and the group generated by the s_α is finite.

Definition 4. A root datum is called *reduced* if $\alpha \in \Phi \Rightarrow 2\alpha \notin \Phi$.

Definition 5. An *isomorphism* of root data $(X, Y, \Phi, \Phi^\vee) \to (X', Y', \Phi', \Phi'^\vee)$ is an isomorphism $X \to X', \Phi \to \Phi'$ whose dual $Y' \to Y, \Phi'^\vee \to \Phi^\vee$.

Note that we did NOT define a "morphism", only "isomorphism".

3.3. **Root system and root datum associated to a pair** (G, T)**.** Suppose G is reductive over perfect field k, $T \leq G$ split maximal torus, $\mathfrak{g}, \mathfrak{t}$ the Lie algebras of G, T respectively, and $\mathrm{Ad} : G \to \mathsf{GL}(\mathfrak{g})$ the adjoint representation. For any character $\alpha \in X^*(T)$, let "α-eigenspace"
$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{Ad}(t)X = \alpha(t)X \quad \forall t \in T(k)\}.$$

Those α such that $\mathfrak{g}_\alpha \neq 0$ are called *roots* of T in G; in which case $\mathfrak{g}_\alpha$ is called *root space*. We denote $\Phi(G, T)$ the set of all roots. Then
$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(G,T)} \mathfrak{g}_\alpha.$$

Fact. Each $\mathfrak{g}_\alpha$ is 1-dimensional.

Thus, for a pair (G, T), we get the associated root system (Φ, V) where $\Phi = \Phi(G, T)$ is the set of roots defined above and $V = \mathit{Span}(\Phi(G, T))$. Then it is known that

Fact. $W(G, T)(k) = W(\Phi, V)$.

Remark. Note that by assumption that G splits, $T \cong \mathbb{G}_m^{rank(T)}$ so $X^*(T) \cong \mathbb{Z}^{rank(T)}$ is a free abelian group and the elements of $\Phi(G, T) \subset X^*(T)$ generates a subgroup H of $\mathbb{Z}^{rank(T)}$ which must also be free abelian; then $V = H \otimes_{\mathbb{Z}} \mathbb{R}$.

Next, there exists a pairing $(,) : V \times V \to \mathbb{C}$ so that elements in Weyl group becomes orthogonal transformation (how?) so for each α, there exists a unique $\alpha^\vee \in X_*(T)$ such that
$$\langle -, \alpha^\vee \rangle = \alpha^\vee(-) = \frac{2(-, \alpha)}{(\alpha, \alpha)} : X^*(T) \to \mathbb{C}.$$

So let $\Phi^\vee := \{\alpha^\vee\}$ and $V^\vee$ likewise defined then $(\Phi^\vee, V^\vee)$ is another root system and
$$\Psi(G, T) := (X^*(T), X_*(T), \Phi, \Phi^\vee)$$
is a root datum.

Example 1. Let us consider $G = \mathsf{GL}_n$ and $T \cong \mathbb{G}_m^n$ being the maximal torus consisting of diagonal matrices. We have seen that $X^*(T) \cong \mathbb{Z}^n \cong X_*(T)$ via identification

$$\left[\chi_{(k_1,...,k_n)} : \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_1^{k_1}...t_n^{k_n}\right] \leftarrow\!\shortmid (k_1,...,k_n) \mapsto \left[t \mapsto \begin{pmatrix} t^{k_1} & & \\ & \ddots & \\ & & t^{k_n} \end{pmatrix}\right]$$

and the pairing $X^*(T) \times X_*(T)$ is the standard inner product.

The roots are characters

$$e_{ij} : \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_i t_j^{-1}$$

for every $i \neq j$ and the corresponding root space

$$\mathfrak{g}_{e_{ij}} = \mathbb{C}(\delta_{k,i}\delta_{l,j})_{k,l}.$$

The co-root $e_{ij}^\vee$ is the one corresponding to $(0, ..., {}^{[i]}1, ..., {}^{[j]}-1, 0, ..., 0) \in \mathbb{Z}^n$. To see this, let $\alpha = \chi_{(k_1,...,k_n)}$ and one checks out

$$\begin{aligned} \mathfrak{g}_\alpha &= \{X \in \mathfrak{g} \mid \mathrm{Ad}(t)X = \alpha(t)X \quad \forall t \in T(k)\} \\ &= \{X \in M_n \mid \mathrm{Diag}(t_1, ..., t_n)X\mathrm{Diag}(t_1^{-1}, ..., t_n^{-1}) = \prod t_i^{k_i} X \quad \forall t_1, ..., t_n \in k\} \\ &= \{X \in M_n \mid t_i t_j^{-1} X_{ij} = \prod t_l^{k_l} X_{ij} \quad \forall 1 \leq i, j \leq n \; \forall t_1, ..., t_n \in k\} \end{aligned}$$

Post-multiplication by $\mathrm{Diag}(t_1^{-1}, ..., t_n^{-1})$ adjusts the columns by the corresponding factor and pre-multiplication by $\mathrm{Diag}(t_1, ..., t_n)$ adjust the row. So suppose that $\mathfrak{g}_\alpha \neq 0$ so we have some $X \neq 0$ in $\mathfrak{g}_\alpha$. Then if $X_{ij} \neq 0$, we must have $t_i t_j^{-1} = \prod t_l^{k_l}$ in a field. So $\alpha = e_{ij}$. It is also clear that the root space $\mathfrak{g}_{e_{ij}}$ consists of matrix whose only allowable non-zero entry is at (i, j).

Example 2. Now consider $G = GU(n, n)$ over imaginary quadratic field $K/\mathbb{Q}$. Does this group have a $\mathbb{Q}$-split maximal torus? Note that $U(n, n)$ does not split i.e. it does not have a $\mathbb{Q}$-split maximal torus.

4. Classification

Theorem of Chevalley-Demazure: Assume $\overline{k} = k$. Then the association $G \mapsto \Psi(G, T)$ is a bijection between

$$\{\text{isom. classes of connected reductive groups } /k\}$$

and

$$\{\text{isom. classes of reduced root data}\}.$$

5. Borel and Parabolic subgroup

Assume G reductive over a perfect field k.

Definition 6. A closed subgroup $B \leq G$ is called a *Borel subgroup* if $B_{\overline{k}}$ is maximal connected solvable subgroup of $G_{\overline{k}}$.

Definition 7. A closed subgroup $P \leq G$ is called a *parabolic subgroup* if $P_{\overline{k}}$ contains a Borel subgroup of G.

Appendix: Geometric Invariant Theory and Quotient

References

[1] James S. Milne. Basic theory of affine group schemes, 2012. Available at www.jmilne.org/math/.

[2] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric Invariant Theory*. Number v. 34 in Ergebnisse der Mathematik und ihrer Grenzgebiete : a series of modern surveys in mathematics. Springer Berlin Heidelberg, 1994.