ALGEBRAIC GROUPS III - LIE ALGEBRAS

LAWRENCE VU

1. Lie algebra

1.1. **Some differential geometry.** The concept of Lie algebra comes from differential geometry of Lie groups i.e. groups that are also ($\mathbb{R}$ or $\mathbb{C}$) differentiable manifolds and such that its group operations are smooth. For notation, recall that there is a functor

$$\mathsf{T} : \textbf{SmoothManifolds} \to \textbf{VectorBundle}$$
$$M \mapsto \mathsf{T}M := \bigsqcup_{p \in M} \mathsf{T}_p M = \{(p, X) \mid p \in M, X \in \mathsf{T}_p M\}$$

For a smooth map $f : M \to N$ between smooth manifolds, we denote $f^* = \mathsf{T}(f)$ and $f^*_x = \mathsf{T}(f)(x) : \mathsf{T}_x M \to \mathsf{T}_{f(x)} N$. A vector field is a smooth section $X : M \to \mathsf{T}M$ where $\mathsf{T}M$ has a canonical smooth structure[1]; we write X_p for $X(p)$.

The Lie algebra of a Lie group G, typically denoted by $\mathfrak{g}$, is just the algebra of left invariant vector fields of G; it comes with a Lie bracket (Lie derivative) defined by[2]

$$\begin{aligned}[X, Y]_g &= \left.\frac{d}{dt}\right|_{t=0} \varphi^*_{X,-t}(Y \circ \varphi_{X,t}) \\ &= \lim_{t \to 0} \frac{\varphi^*_{X,-t} Y_{\varphi_X(t,g)} - Y_g}{t} \\ &= XY - YX\end{aligned}$$

where $\varphi_X : \mathbb{R} \times G \to G$ is the 1-parameter subgroup/flow associated to X parameterized by G; we write $\varphi_{X,t} = \varphi_X(t, -) : G \to G$ for the induced smooth function. TODO: Check the type correctness here! Note that for X to be left invariant, $X_g = \lambda^*_g(X_e)$ for all $g \in G$ (where $\lambda_g : G \to G; h \mapsto gh$ is left translation by g; we likewise denote ρ_g for the right translation by g) so X is completely determined by its image X_e at the identity $e \in G$. In other words, the Lie algebra can be identified with the tangent space at e

$$\mathfrak{g} = \mathsf{T}_e G \cong \mathbb{R}^{\dim G} \text{ or } \mathbb{C}^{\dim G}$$

so it is a free $\mathbb{R}$ (or $\mathbb{C}$) module of rank $\dim G$. It is easy to check that if X, Y are left invariant then so is $[X, Y]$. Thus, the vector field $[X, Y]$ is completely determined by $[X, Y]_e = \frac{d}{dt}\big|_{t=0} \exp^*_{-tX}(Y \circ \exp_{tX})$ where $\exp_X = \varphi_X(1, e) : \mathfrak{g} \to G$ is the exponential map.

For each $g \in G$, let $\Psi_g = \rho_{g^{-1}} \circ \lambda_g = \lambda_{g^{-1}} \circ \rho_g : G \to G; h \mapsto ghg^{-1}$ be the natural inner automorphism of G (obviously smooth). Then G acts naturally on its Lie algebra $\mathfrak{g}$ by $g \cdot X = (\Psi^*_g)X = \rho^*_{g^{-1}} \lambda^*_g X$. This *smooth* action is denoted by

$$\mathrm{Ad} : G \to \mathsf{GL}(\mathfrak{g}).$$

Note that $\mathsf{GL}(\mathfrak{g})$ is also a Lie group. Taking derivative of this smooth map at identity, we get

$$\mathrm{ad} = \mathrm{Ad}^*_e : \mathsf{T}_e G \to \mathsf{T}_I \mathsf{GL}(\mathfrak{g})$$

[1] So that trivial projection map $\mathsf{T}M \to M$ is smooth.

[2] Basically we take derivative of Y along the integral curve of X. But since Y_{φ_t} and $Y_{\varphi_0} = Y_g$ are in different tangent spaces, we pull Y_{φ_t} back so we can subtract.

and the Lie bracket can be recovered as[3]

$$[X,Y] = \mathrm{ad}(X)(Y).$$

I think this is like embedding $G \to \mathsf{T}G$ so that we can take second derivative in G whence $XY - YX$ makes sense.

1.2. **Abstract Lie algebra and the Lie functor.** Now we do everything abstractly. Let k be a ring. The references are [1], chater XI and [2], chapter 12.

Definition 1. A *Lie algebra* over k is a *free* k-module of finite rank A together with a bracket $[,] : A \times A \to A$ satisfying

(i) $[X,X] = 0$, and
(ii) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ for all $X, Y, Z \in A$.

Example 1. The matrix algebra $A = M_n(k)$ together with the bracket defined by $[X,Y] = XY - YX$ is a Lie algebra. When $k = \mathbb{R}$ or $k = \mathbb{C}$, we get the Lie algebra of the Lie group $\mathsf{GL}_n(\mathbb{R})$ or $\mathsf{GL}_n(\mathbb{C})$.

Our goal here is to define a functor

$$\mathrm{Lie} : \mathbf{AffAlgGroup}_k \to \mathbf{LieAlg}_k$$

where if G be an affine algebraic group then as a set[4],

$$\mathrm{Lie}\,(G) := \ker G(f)$$

Here, $D_k := k[t]/(t^2)$ is the ring of dual numbersand the map

$$G(f) : G(D_k) \to G(k)$$

is the corresponding group homomorphism

$$f : D_k \to k \qquad [t] \mapsto 0$$

under functoriality of G. We first need to make the set $\mathrm{Lie}\,(G)$ into a k-algebra and then we define the bracket.

Let $A := \mathcal{O}(G)$ be the representing ring for G, $\epsilon : A \to k$ be the co-identity ring homomorphism and $I = \ker(\epsilon)$. (See my first note for the example for GL_n.) The idea is that there is a canonical bijection

$$\mathrm{Lie}\,(G) \leftrightarrow \mathrm{Hom}_k(I/I^2, k)$$

and the latter already has natural k-module structure. (Note that despite being called "algebra", there is no natural multiplication in a Lie algebra. Certainly, one can add elements of $\mathrm{Hom}_k(I/I^2, k)$ or multiply them with elements of k.)

[3]This is essentially a tautology i.e. straight-forward from the definition. To compute $\mathrm{Ad}_e^*(X)$, we take the flow $\exp(tX)$ in G and take derivative of the image curve $\mathrm{Ad}(\exp(tX))$ in $\mathsf{GL}(\mathfrak{g})$ at $t = 0$. Now, by definition, $\mathrm{Ad}(\exp_{tX}) : Y \mapsto \rho^*_{\exp(-tX)}\lambda^*_{\exp(tX)}Y$ and so taking derivative

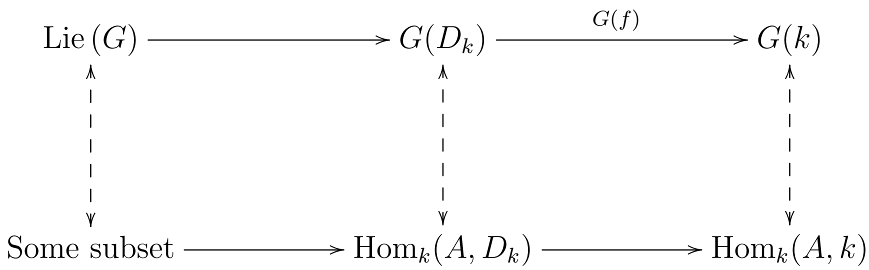
$$\begin{aligned}\frac{d}{dt}\mathrm{Ad}(\exp_{tX}) : Y &\mapsto \frac{d}{dt}\rho^*_{\exp(-tX)}\,\lambda^*_{\exp(tX)}\,Y \\ &= \rho^*_{\exp(-tX)}\,Y_{\exp(tX)} && \text{since } Y \text{ is left invariant} \\ &= [X,Y]_e\end{aligned}$$

by previous definition.

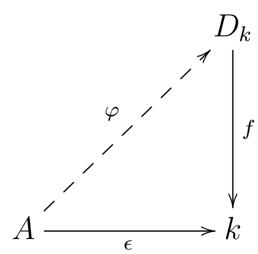
[4]According to [2], this is the analogy of the tangent space at e.

Remark. A curious reader should ask the question "Why don't we use this as the definition instead?". My best guess for reason is that the original definition is the direct analogy with our earlier discussion on classical Lie group: The Lie algebra is identified with tangent space at identity; think of $G(D_k)$ as the tangent bundle $\mathsf{T}G$ and taking the kernel (pre-image of the identity of $G(k)$) as extracting the section at identity of the left-invariant vector field over G.

To see the bijection, note that $G(R) \leftrightarrow \mathrm{Hom}_k(A, R)$ for any R by representability. In particular, we have the commutative diagram



where we can see that elements of $\mathrm{Lie}\,(G)$ can be identified with a natural subset of $\mathrm{Hom}(A, D_k)$ whose images in $\mathrm{Hom}_k(A, k)$ is the identity element of $\mathrm{Hom}_k(A, k)$; which should obviously be ϵ. Note that the map $\mathrm{Hom}(A, D_k) \to \mathrm{Hom}_k(A, k)$ is obtained by applying the functor $\mathrm{Hom}_k(A, -)$ to the morphism $f : D_k \to k$ so it is nothing but $\varphi \mapsto f \circ \varphi$. Thus, to sum up to this point, elements of $\mathrm{Lie}\,(G)$ are in one-to-one correspondence with k-algebra homomorphism $A \to D_k$ whose post-composition with f is the co-identity ϵ. In other words, $\mathrm{Lie}\,(G)$ consists of all the φ we can fit into the diagram (in $\mathbf{Alg}_k$)



This diagram implies $\varphi(I) \subseteq \ker(f) = (t)$. As a result, $\varphi(I^2) \subseteq (t)^2 = 0$ i.e. $I^2 \subseteq \ker\varphi$ so φ descends to (is completely determined as) a homomorphism $A/I^2 \to D_k$. Now $A/I^2 \cong k \oplus I/I^2$ by $[a] \mapsto (\epsilon(a), [a - \epsilon(a)])$ so a morphism $A/I^2 \to D_k$ is completely determined by its restriction to $I/I^2 \to k$.

Exercise 1. Why is $\mathrm{Hom}_k(I/I^2, k)$ a free k-module? Why is it of finite rank?

Exercise 2. Describe morphism $\mathrm{Lie}\,(\psi) : \mathrm{Lie}\,(G) \to \mathrm{Lie}\,(H)$ corresponding to morphism of affine algebraic groups $\psi : G \to H$.

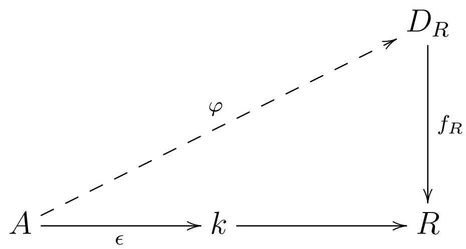
(Note that there are two viewpoints on Lie. In the original definition, I expect that $\mathrm{Lie}\,(\psi)$ is just the restriction of $\psi_{D_k} : G(D_k) \to H(D_k)$ to the subgroup $\mathrm{Lie}\,(G) \subset G(D_k)$. Then as we identify $\mathrm{Lie}\,(G)$ with $\mathrm{Hom}_k(I/I^2, k)$, I expect that the map $\mathrm{Hom}_k(I/I^2, k) \to \mathrm{Hom}_k(J/J^2, k)$ is just by composition with ψ. Here, J is the kernel of co-identity for H.)

To get the bracket, we replicate differential geometry; the only difference is that we need to "instantiate the algebraic group to get actual groups". We have a linear representation[5]

$$\mathrm{Ad} : G \to \mathsf{GL}_{\mathrm{Lie}\,G}$$

[5]Note: Ad must be a natural transformation of functors; in other words, for each $R \in \mathbf{Alg}_k$, we have a group homomorphism $\mathrm{Ad}_R : G(R) \to \mathsf{GL}_{\mathrm{Lie}\,G}(R)$ and each morphism $R \to R'$, a certain diagram commutes.

where for each k-algebra R, recall that $(\mathrm{Lie}\, G) \otimes R$ consists of maps that can be fit into the diagram



so we can view $\mathrm{Lie}\, G \otimes R$ as a subset of $G(D_R) \leftrightarrow \mathrm{Hom}_k(A, D_R)$ so that $G(D_R)$ acts naturally on $\mathrm{Lie}\, G \otimes R$ by conjugation[6]. The canonical homomorphism $h_R : R \to R[t]/(t^2)$ gives rise to a group homomorphism $G(h_R) : G(R) \to G(D_R)$ with which we obtain an action of $G(R)$ on $\mathrm{Lie}\, G \otimes R$.

Exercise 3. Check that this action preserves the Lie algebra. (Obvious since kernel is stable under conjugation.)

Now, $\mathsf{GL}_{\mathrm{Lie}\, G} = \mathsf{GL}_r$ is an affine algebraic group over k since $\mathrm{Lie}\, G \cong k^r$ is free k-module of finite rank r so we can apply the partially defined functor Lie to the morphism Ad of affine algebraic groups to get a morphism

$$\mathrm{ad} = \mathrm{Lie}\,(\mathrm{Ad}) : \mathrm{Lie}\,(G) \to \mathrm{Lie}\,(\mathsf{GL}_{\mathrm{Lie}\, G})$$

of k-algebra. Finally, we can define the bracket

$$[X, Y] := \mathrm{ad}(X)(Y).$$

Exercise 4. Understand identification of $\mathrm{ad}(X)(Y)$ with an element of $\mathrm{Lie}\,(G)$.

Example 2. We check the case $G = \mathsf{GL}_n$. A matrix in $G(D_k) = \mathsf{GL}_n(D_k)$ can be written as $X + [t]Y$ with $X, Y \in M_n(k)$. The map

$$\mathsf{GL}_n(D_k) \to \mathsf{GL}_n(k)$$

is

$$X + [t]Y \mapsto X$$

so

$$\begin{aligned} \mathrm{Lie}\,(G) &= \ker(G(D_k) \to G(k)) \\ &= \{X + [t]Y \in \mathsf{GL}_n(D_k) \mid X = I_n\} \\ &= \{I_n + [t]Y \mid Y \in M_n(k)\}. \end{aligned}$$

(Note that the last line requires checking that all matrices of the form $I_n + [t]Y$ are invertible.)

So $\mathrm{Lie}\,(\mathsf{GL}_n) \cong M_n(k) \cong k^{n^2}$.

Now we compute the bracket. Recall that the representing ring for GL_n is

$$A = \mathcal{O}(\mathsf{GL}_n) = k[x_{ij}, y]/(\det(x_{ij})y - 1)$$

and the co-identity

$$\epsilon : A \to k$$

is given by $x_{ij} \mapsto \delta_{ij}$ and $[y] \mapsto 1$. Evidently, one has

$$I = \ker(\epsilon) = (x_{ij} - \delta_{ij}, y - 1)$$

i.e. the ideal of A generated by the elements $x_{ij} - \delta_{ij}$ and

$$I^2 = ((x_{ij} - \delta_{ij})(y - 1), (x_{ij} - \delta_{ij})(x_{kl} - \delta_{kl}), (y - 1)^2)$$

[6]Unfortunately, we can't define the action explicitly. The problem is that we need to identify the map in $\mathrm{Lie}\, G \otimes R$ back to the abstract group elements in $G(R)$ to do the conjugation.

so I/I^2 is free k-module with basis $\{x_{ij} - \delta_{ij}\}$ of n^2 elements which is isomorphic to its dual $\mathrm{Hom}_k(I/I^2, k)$. Thus we have

$$\mathrm{Lie}\,(\mathsf{GL}_n) \leftrightarrow \mathrm{Hom}_k(I/I^2, k) \cong k^{n^2}.$$

The representation

$$\mathrm{Ad} : \mathsf{GL}_n \to \mathsf{GL}(\mathrm{Lie}\,(\mathsf{GL}_n)) \cong \mathsf{GL}_{n^2}$$

can be given explicitly: For each k-algebra R, one first identify $\mathrm{Lie}\,(\mathsf{GL}_n) \otimes R \cong M_n(R)$ as subset of $\mathrm{Hom}(A, D_R)$ where a matrix $Y \in M_n(R)$ is identified with the map

$$\varphi_Y : A \to D_R; x_{ij} \mapsto \delta_{ij} + [t]Y_{ij}$$

which can be viewed as invertible matrix $(\delta_{ij} + [t]Y_{ij})$ in $\mathsf{GL}_n(D_R)$ by representability so that any invertible matrix $g \in \mathsf{GL}_n(R)$, we define the action

$$g \cdot Y = \psi_g^{-1} \varphi_Y \psi_g$$

where $\psi_g = g \in \mathsf{GL}_n(D_R)$ but viewed as a map $\psi_g : A \to D_R$ by $\psi_g(x_{ij}) = g_{ij}$. The right hand side is product in $\mathsf{GL}_n(D_R)$. Of course, for this to be a genuine group action, the result must have interpretation in $\mathrm{Lie}\,(\mathsf{GL}_n) \otimes R \cong M_n(R)$. In other words, we claim that

$$g(I_n + [t]Y)g^{-1} \equiv I_n \bmod (t^2)$$

which is clear since

$$g(I_n + [t]Y)g^{-1} = gI_ng^{-1} + g[t]Yg^{-1} = I_n + [t]g^{-1}Yg$$

by standard property of matrix multiplication. It is also clear from this computation that this is indeed the familiar conjugation of GL_n on M_n.

Finally, we obtain

$$\mathrm{ad} : \mathrm{Lie}\,(\mathsf{GL}_n) \to \mathrm{Lie}\,(\mathsf{GL}_{n^2})$$

by applying Lie to Ad; which is explicitly just the restriction of Ad on the kernel:

$$\begin{array}{ccccc} \mathrm{Lie}\,\mathsf{GL}_n & \longrightarrow & \mathsf{GL}_n(D_k) & \longrightarrow & \mathsf{GL}_n(k) \\ \big\downarrow \mathrm{ad} & & \big\downarrow \mathrm{Ad}_{D_k} & & \big\downarrow \mathrm{Ad}_k \\ \mathrm{Lie}\,\mathsf{GL}_{n^2} & \longrightarrow & \mathsf{GL}_{n^2}(D_k) & \longrightarrow & \mathsf{GL}_{n^2}(k) \end{array}$$

and so we recover the familiar

$$[X, Y] = \mathrm{ad}(X)(Y) = XY - YX.$$

To see that, by definition, Ad_{D_k} is obtained by embedding $\mathsf{GL}_n(D_k) \to \mathsf{GL}_n(D_{D_k}) = \mathsf{GL}_n(k[s,t]/s^2, t^2); X_1 + tX_2 \mapsto X_1 + tX_2$ to let it acts on

$$\begin{aligned} \mathrm{Lie}\,\mathsf{GL}_n \otimes D_k &= \mathrm{Lie}\,\mathsf{GL}_{n/D_k} \subset \mathsf{GL}_n(D_{D_k}) \\ &= \{I_n + sY \mid Y \in M_n(D_k)\} \\ &= \{I_n + s(Y + tZ) \mid Y, Z \in M_n(k)\} \end{aligned}$$

by conjugation. In particular, $I_n + tX \in \mathrm{Lie}\,(G) \subset \mathsf{GL}_n(D_k)$ action is given by

$$\begin{aligned} I_n + s(Y + tZ) &\mapsto (I_n + tX)(I_n + s(Y + tZ))(I_n - tX) \\ &= I_n + s(I_n + tX)Y(I_n - tX) + st(I_n + tX)Z(I_n - tX) \\ &= I_n + s((Y + tXY) - (Y + tXY)tX) + st((Z + tXZ) - t(Z + tXZ)X) \\ &= I_n + sY + stXY - stYX + stZ \\ &= I_n + s(Y + tXY - tYX + tZ) \end{aligned}$$

$$= I_n + s(Y + tZ + t(XY - YX))$$

since $(I_n + tX)^{-1} = I_n - tX$. To sum up:

$$\begin{aligned} \mathrm{Ad}(I_n + tX) : \mathrm{Lie}\,\mathsf{GL}_{n/D_k} &\to \mathrm{Lie}\,\mathsf{GL}_{n/D_k} \\ Y + tZ &\mapsto Y + tZ + t(XY - YX) \end{aligned}$$

View $Y + tZ + t(XY - YX)$ as the pair of $(Y + tZ, XY - YX)$ and you see that $\mathrm{Ad}(I_n + tX)$ acts as $I_n + t(XY - YX)$. Thus, one extracts the bracket as claimed. See also Lemma 8.2 in [1]. (I should have put $Z = 0$ and then we see that the transformation is $Y \mapsto Y + t(XY - YX)$ so clearly the transformation is $I + t(XY - YX)$.

1.3. **Other properties.** The functor Lie is left exact, commutes with fiber product.

References

[1] James S. Milne. Basic theory of affine group schemes, 2012. Available at www.jmilne.org/math/.
[2] James S. Milne. Algebraic groups (v2.00), 2015. Available at www.jmilne.org/math/.