ALGEBRAIC GROUPS II - BASIC CONSTRUCTIONS

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Our goal is to import various notions in group theory such as (normal) subgroup into the theory of algebraic groups. Since there are three definitions of algebraic groups, we likewise have various way to define these concepts.

1. Recall: The category of group schemes

Let $\mathfrak{G} = (G, m_G, e_G, i_G)$ and $\mathfrak{H} = (H, m_H, e_H, i_H)$ be group schemes over S. A group scheme homomorphism $\phi : \mathfrak{G} \to \mathfrak{H}$ is a morphism of the underlying schemes $G \to H$ such that the diagrams in category $\mathbf{Scheme}_S$ (you guess it)

$$\begin{array}{ccc} G \times G & \xrightarrow{m_G} & G \\ \downarrow{\scriptstyle \phi\times\phi} & & \downarrow{\scriptstyle \phi} \\ H \times H & \xrightarrow{m_H} & H \end{array}$$

and

$$\begin{array}{ccc} G & \xrightarrow{e_G} & * \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{Id}} \\ H & \xrightarrow{e_H} & * \end{array}$$

commutes. With this definition of morphism, we have a category $\mathbf{GroupScheme}_S$ of group schemes over S.

In the functorial definition, a morphism $\phi : \mathfrak{G} \to \mathfrak{H}$ is just a natural transformation of functors. Concretely, for each S-scheme T, ϕ gives us a group homomorphism $\phi_T : \mathfrak{G}(T) \to \mathfrak{H}(T)$.

Likewise, one easily comes up with a definition of morphisms between Hopf algebras.

Exercise 1. Prove that the three definitions of morphisms are equivalent.

Exercise 2 (Theorem 3.1 of [1], a criterion for representability)**.** Let $\mathfrak{G} : k\text{-}\mathbf{Alg} \to \mathbf{Sets}$ be a functor. If $\mathfrak{G}$ is representable, then for every faithfully flat homomorphism $R \to R'$ of k-algebras, the sequence

$$\mathfrak{G}(R) \to \mathfrak{G}(R') \rightrightarrows \mathfrak{G}(R' \otimes_R R')$$

is exact (i.e. the image of $\mathfrak{G}(R)$ is the equalizer of the following two maps). Conversely, if there exists a faithfully flat homomorphism $k \to k'$ such that (a) $\mathfrak{G}|_{k'\text{-}\mathbf{Alg}}$ is representable, and (b) for all k-algebras R, the sequence $\mathfrak{G}(R) \to \mathfrak{G}(R_{k'}) \rightrightarrows \mathfrak{G}(R_{k'} \otimes R_{k'})$ is exact then $\mathfrak{G}$ is representable.

2. Kernel group scheme

Now let us view $\mathfrak{G}, \mathfrak{H}$ as functors and let $\phi : \mathfrak{G} \to \mathfrak{H}$ be a morphism. Then we have a natural kernel functor

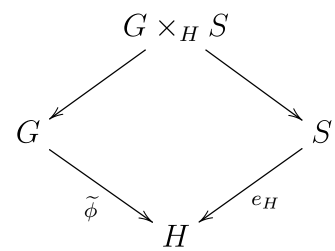
$$\ker_\phi : \mathbf{Scheme}_S^{\mathrm{op}} \to \mathbf{Groups}$$

associated to ϕ where

$$\ker_\phi(T) := \ker(\mathfrak{G}(T) \to \mathfrak{H}(T)).$$

Theorem 1. *The functor* $\ker_\phi$ *is a group scheme.*

Proof. Let G and H be the representing schemes of $\mathfrak{G}$ and $\mathfrak{H}$ respectively and let $\widetilde{\phi} : G \to H$ be the induced morphism of schemes. It is easy to see that the fiber product (in the category of schemes)



represents $\ker_\phi$. ☐

3. Subgroup scheme

In general category theory:

- A morphism f is called a *monomorphism* if it is left-cancellable i.e. for any morphisms g and h, if $f \circ g = f \circ h$ then $g = h$.
- The dual notion is *epimorphism*.
- A sub-object of an object A is an equivalence class of monomorphism $X \to A$ where we say that the monomorphism f and g are equivalent if they factor through each other.

So we define

Definition 1. A group scheme $\mathfrak{H}$ together with a monomorphism $\mathfrak{H} \to \mathfrak{G}$ is called a *subgroup scheme* of $\mathfrak{G}$.

In the functorial definition, this is to say that there is natural transformation of functors $\mathfrak{H} \to \mathfrak{G}$ such that $\mathfrak{H}(T) \to \mathfrak{G}(T)$ are always subgroups. A *normal subgroup scheme* $\mathfrak{H} \to \mathfrak{G}$ is one such that $\mathfrak{H}(T) \to \mathfrak{G}(T)$ is normal for all T. (I don't know the algebraic geometry definition.)

We are interested in *closed subgroup schemes* i.e. one where the underlying scheme is closed subscheme. In the affine case, this is to say that $\mathcal{O}(H) = \mathcal{O}(G)/I$. For example, SL_n is a closed algebraic subgroup of GL_n which in turn is closed algebraic subgroup of SL_{n+1}.

4. Quotient

Suppose that $\mathfrak{H} \to \mathfrak{G}$ is normal subgroup schemes; both over $\mathrm{Spec}(k)$ so we can view as functors k-**Alg** $\to$ **Groups**. We can define the functor

$$\mathfrak{G}/\mathfrak{H} : k\text{-}\mathbf{Alg} \to \mathbf{Groups}$$
$$A \mapsto \mathfrak{G}(A)/\mathfrak{H}(A)$$

For it to be an affine group scheme (resp. affine algebraic group), we need a k-algebra R (resp. k-algebra R of finite type) such that $\mathfrak{G}/\mathfrak{H}(A) = \mathrm{Hom}(R, A)$ for all A. It is not clear how one would get such an object from the corresponding representation for $\mathfrak{G}$ and $\mathfrak{H}$.

According to Wikipedia: *"For a subgroup scheme H of a group scheme G, the functor that takes an S-scheme T to $G(T)/H(T)$ is in general not a sheaf, and even its sheafification is in general not representable as a scheme. However, if H is finite, flat, and closed in G, then the quotient is representable, and admits a canonical left G-action by translation. If the restriction of this action to H is trivial, then H is said to be normal, and the quotient scheme admits a natural group law. Representability holds in many other cases, such as when H is closed in G and both are affine (Raynaud, Michel (1967), Passage au quotient par une relation d'quivalence plate)."*

Example 1. SL_n is normal subgroup scheme of GL_n since it is kernel of $\det : \mathsf{GL}_n \to \mathsf{GL}_1$. The quotient we know is the group scheme GL_1.

Exercise 3. Find an algorithm to find [the representing scheme] quotient of group schemes. In other words, given affine group schemes $\mathfrak{G} = \operatorname{Spec}(A)$ and closed subgroup $\mathfrak{H} = \operatorname{Spec}(B)$, find the Hopf algebra C that represents $\mathfrak{G}/\mathfrak{H}$.

5. Base change

If S' is a scheme over S and $\mathfrak{G}$ is a group scheme over S then we can define a functor $\mathfrak{G}_{S'}$ given by

$$\mathfrak{G}_{S'}(T) = \mathfrak{G}(T_S)$$

for any S'-scheme T. Here, T_S is literally the scheme T but treated as a scheme over S in trivial way i.e. via composition $T \to S' \to S$.

It is easy to see that if G represents $\mathfrak{G}$ then $\mathfrak{G}_{S'}$ is represented by $G \times_S S'$ by universal property of fiber product. Thus, $\mathfrak{G}_{S'}$ is a group scheme over S'.

6. Restriction of scalars

Let k' be a k-algebra. For a group scheme $\mathfrak{G}$ over k', we have a restriction of scalar functor ($\mathsf{Res}_{k'/k}$ is a function on functor and give a functor as output)

$$\begin{aligned} \mathsf{Res}_{k'/k}\mathfrak{G} : k\textbf{-Alg} &\to \textbf{Groups} \\ A &\mapsto \mathfrak{G}(k' \otimes_k A) \end{aligned}$$

Similar to the situation of quotient, for $\mathsf{Res}_{k'/k}\mathfrak{G}$ to be a group scheme, we need a scheme that represents it and there is no clear way of getting such a scheme. (See example below to see illustration of the complications.) This can be guaranteed in case such as k' is finitely generated and projective as a k-module, according to Proposition 5.1 in [1], page 60.

Example 2. Let $k/\mathbb{Q}$ be an imaginary quadratic field (or any totally complex field i.e. imaginary quadratic extension k/F of a totally real field F; we only need an involution). We define the unitary similitude group functor

$$\mathsf{U}_k(n, n) : \mathbb{Z}\textbf{-Alg} \mapsto \textbf{Groups}$$

by

$$\mathsf{U}_k(n, n)(A) := \{g \in M_{2n}(\mathcal{O}_K \otimes A) \mid g^* J_{2n} g = J_{2n}\}$$

and $J_{2n} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ where g^* is conjugate transpose[1]. I want to emphasize the fact that this functor depends on k. Also note that this is not an algebraic group over $\mathcal{O}_K$ because complex conjugation is not an algebraic operation.

[1]Observe that every $\mathbb{Z}$-algebra A, the $\mathcal{O}_K$-algebra $\mathcal{O}_K \otimes A$ has conjugation induced by the conjugation of k, namely on basic tensor $\overline{x \otimes a} = \overline{x} \otimes a$.

To make it into an algebraic group, we need a scheme over $\mathbb{Z}$ that represents it and we make use of restriction of scalar:

$$\mathsf{U}_k(n,n) = \{g \in \mathsf{Res}_{\mathcal{O}_K/\mathbb{Z}} \mathsf{GL}_{2n/\mathcal{O}_K} \mid g^* J g = J\}$$

which exists since $\mathcal{O}_K/\mathbb{Z}$ is finitely generated and projective (in fact, free of rank 2).

To write down the scheme explicitly, recall that $\mathcal{O}_K = \mathbb{Z} \oplus \mathbb{Z}\omega$ where ω is root of some quadratic polynomial $F \in \mathbb{Z}[X]$ and note that $\overline{\omega} \in \mathcal{O}_K$ so it can be expressed as $a + b\omega$ for some fixed integer $a, b \in \mathbb{Z}$. A matrix $g \in M_{2n}(\mathcal{O}_K \otimes A)$ can be broken as $g = 1 \otimes X + \omega \otimes Y$ with X, Y having entries in A. Then $g^* = 1 \otimes X^t + \overline{\omega} \otimes Y^t = 1 \otimes X^t + (a + b\omega) \otimes Y^t$. Now let

$$R = \mathbb{Z}[x_{ij}, y_{ij}, w]/(F(w), g^* J_{2n} g - J_{2n})$$

where $g^* J_{2n} g - J_{2n}$ consists of $(2n)^2$ equations.

From this example, one can see that restriction of scalars has the effect of "adding additional equations to define parameters in k' over k".

7. Intersection

Proposition 1.36 of [2]: Let H_j be a family of algebraic subgroups of G. Then $H := \bigcap H_j$ is an algebraic subgroup of G. If G is affine, then H is affine, and its coordinate ring is $\mathcal{O}(G)/I$ where I is the ideal in $\mathcal{O}(G)$ generated by the ideals $I(H_j)$ of the H_j.

8. Center and derived subgroup

Let G be algebraic group (viewed as a functor) over k and H an algebraic subgroup then the functor $N_G(H)$ where

$$N_G(H)(R) := \{g \in G(R) \mid gH(R)g^{-1} = H(R)\}$$

is an algebraic subgroup (c.f. [2], Proposition 1.59). Likewise, the functor

$$C_G(H) : R \mapsto \{g \in G(R) \mid g \text{ centralizes } H(R') \text{ in } G(R') \text{ for all } R-\text{algebra } R'\}$$

is an algebraic group by Proposition 1.67 of [2]. We define the center of G to be $Z_G := C_G(G)$.

When k is a field, we define the derived subgroup G^{der} to be the intersection of all[2] normal subgroups N of G such that G/N is commutative. By Proposition 8.20 of [2], G^{der} is generated by commutators map $(x, y) \mapsto xyx^{-1}y^{-1}$ when G is affine or smooth. (Note that if K is a group, the set of commutators $\{xyx^{-1}y^{-1}\}$ needs not be a group though it is closed under inverse and has identity so we have to take the group generated by it to get a subgroup of K. In other words, $[K, K] = \{x_1y_1x_1^{-1}y_1^{-1}...x_ry_rx_r^{-1}y_r^{-1}\}$ is a subgroup of K.) It is easy to find out explicitly the Hopf algebra in affine case. We illustrate that in the following

Example 3. Abstractly, the center of $\mathsf{GL}_n(A)$ consists of aI_n for $a \in A^\times$ and the derived subgroup of $\mathsf{GL}_n(A)$ is precisely $\mathsf{SL}_n(A)$. To construct the derived subgroup of GL_n in abstract way, recall the Hopf algebra for GL_n is $A = k[x_{ij}, y]/(\det(x_{ij})y - 1)$ and co-multiplication and co-inverse are as given before. Set $A^r = A \otimes A \otimes \cdots \otimes A$ and we get the co-commutator map $c_r : A \to A^{2r}$ (basically formula to compute commutator from usual matrix multiplication). Let $I_r := \ker(c_r)$ and $I := \bigcap I_r$. Then GL_n^{der} is represented by A/I.

References

[1] James S. Milne. Basic theory of affine group schemes, 2012. Available at www.jmilne.org/math/.
[2] James S. Milne. Algebraic groups (v2.00), 2015. Available at www.jmilne.org/math/.

[2] This is a finite intersection since k is a field and so the scheme representing G is noetherian.