ALGEBRAIC GROUPS I – THREE DEFINITIONS

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The goal of this very first note is to give 3 equivalent definitions of (affine) algebraic groups over a ring k. All rings are commutative with 1.

(i) As group object in category of scheme;
(ii) As a Hopf k-algebra;
(iii) As a representable functor k-**Alg** $\to$ **Groups**.

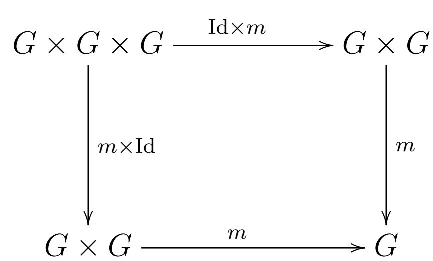
We assume the reader is in full mastery of algebraic geometry; such as complete command[1] of Hartshorne [2] or Grothendieck's EGA [1]. My main reference is Milne's books [3], [4].

1. Group Scheme

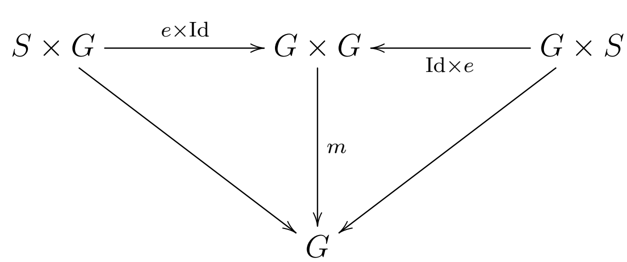
Let S be a fixed scheme; for example $S = \mathrm{Spec}(k)$. Hereafter, if X and Y are S-schemes then $X \times Y$ means the fiber product $X \times_S Y$ in the category of S-schemes.

Definition 1. A *group scheme* over S is a tuple (G, m, e, i) where G is an S-scheme and $m : G \times G \to G, e : S \to G$ and $i : G \to G$ are morphisms of S-schemes satisfying the familiar group axioms in the form of commutative diagrams[2]:

(i) Associativity:

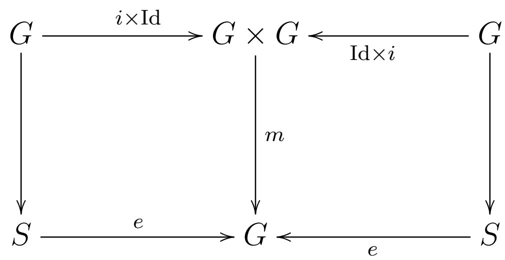


(ii) Property of unit element of G:



[1]This is a joke.
[2]Think of S as the trivial group scheme. We will see the reason later in the third viewpoint.

(iii) Existence of inverse:



An *algebraic group* over S is a group scheme over S of finite type. An *affine algebraic group* is an algebraic group whose underlying scheme is affine i.e. $G = \text{Spec}(A)$ for some ring A.

Example 1. An elliptic curve E defined over a field k should be an algebraic group over $S = \text{Spec}(k)$. Unfortunately, it is typically hard and time consuming to write down a morphism between schemes so we won't give the explicit m_E, e_E and i_E.

Example 2. Standard algebra and algebraic geometry taught us that groups whose operations are "polynomials" should be algebraic group since they are group in category of algebraic varieties. With that the general linear group GL_n should be an algebraic group; but as with elliptic curve, it is hard to write down the scheme morphisms explicitly. The underlying scheme is easy though

$$G = \text{Spec}(k[x_{ij}, y]/(\det(x_{ij})y - 1)).$$

It is not hard to guess the definition of morphism of group schemes; so we get a natural category of group schemes over S, denoted by $\textbf{GroupScheme}_S$, as well as its subcategory of algebraic groups.

2. Affine algebraic group as Hopf algebra

Recall from [2] the following facts:

(i) Giving a morphism of schemes $X \to Y$ where Y is affine is equivalent to giving a homomorphism of rings $\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$. As a consequence, the categories of affine schemes and commutative rings with 1 are contra-equivalent; more precisely, the Spec-functor gives an equivalent of categories.

(ii) If $X = \text{Spec}(A), Y = \text{Spec}(B)$ are affine schemes over the affine scheme $S = \text{Spec}(k)$ then $X \times_S Y \cong \text{Spec}(A \otimes_k B)$.

With this two facts, we see that giving an affine group scheme G over $\text{Spec}(k)$ is equivalent to giving a k-algebra $A = \Gamma(G) = \mathcal{O}(G)$ together with ring homomorphisms $\mu : A \to A \otimes_k A$, $\epsilon : A \to k$ and $\iota : A \to A$, that is the *co-multiplication, co-identity and co-inverse* corresponding to the scheme morphisms appearing in the definition of group schemes), satisfying the 3 commutative diagrams of Definition 1 with A replaced G, k replaced S and all arrow reversed. We call such an algebra a *Hopf algebra*. An affine algebraic group now become a Hopf algebra that is finitely generated over k i.e. is a quotient $k[x_1, ..., x_n]/I$.

Example 3. With this view point, we can complete the structure of GL_n. The Hopf algebra A should be $k[x_{ij}, y]/(\det(x_{ij})y - 1)$ that we gave above. The co-identity[3] map $\epsilon : A \to k$ should simply send $x_{ij} \mapsto \delta_{ij} \in k, y \mapsto 1$, the co-inverse map is simply $x_{ij} \mapsto yM_{ij}, y \mapsto \det(x_{ij})$ where M_{ij} is the appropriate minor and finally, the co-multiplication $\mu : A \to A \otimes_k A = k[X_{ij}, Y, Z_{ij}, W]/(\det(X_{ij})Y - 1, \det(Z_{ij})W - 1)$ sends $x_{ij} \mapsto \sum X_{ik} \otimes Z_{kj}$ and $y \mapsto YW$. Intuitively, these maps tell us how to do the group operations in GL_n.

[3]Recall that to give a map from $k[x_1, ..., x_n]/I \to B$ (where B is any k-algebra) is equivalent to giving a map $k[x_1, ..., x_n] \to B$ such that $I \to 0$.

Note that GL_n depends on k so normally, we should write $\mathsf{GL}_{n/k}$.

Example 4. Write down the definition of morphism of Hopf algebra that corresponds to morphism of affine algebraic groups. Verify the first definition.

3. Algebraic group as representable functors

3.1. **Representable functor.** Let $\mathcal{C}$ be a *locally small* category (i.e. such that $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set for any objects $A, B \in \mathcal{C}$) so that for any fixed object A, we have a natural functor

$$\mathrm{Hom}(A, -) : \mathcal{C} \to \mathbf{Sets}$$

For any object $B \in \mathcal{C}$, $\mathrm{Hom}(A, B)$ should be the obvious set. As for each morphism $f : B \to B'$ in $\mathcal{C}$, the corresponding morphism $\mathrm{Hom}(A, -)(f) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B')$ is given by $g \mapsto f \circ g$. See appendix for proof that this is a functor.

Definition 2. A functor $\mathfrak{F} : \mathcal{C} \to \mathbf{Sets}$ is *representable* if it is naturally isomorphic to $\mathrm{Hom}(A, -)$ for some object $A \in \mathcal{C}$.

In other words, there exists a *representation* (A, Φ) for $\mathfrak{F}$ where

$$\Phi : \mathrm{Hom}(A, -) \to \mathfrak{F}$$

is a natural isomorphism of functors; in other words, for any $B \in \mathcal{C}$, Φ gives us a bijection $\Phi(B) : \mathrm{Hom}(A, B) \to \mathfrak{F}(B)$ between two sets so that the diagram

$$\begin{array}{ccc}
\mathrm{Hom}(A, B) & \xrightarrow{\Phi(B)} & \mathfrak{F}(B) \\
\Big\downarrow{\scriptstyle \mathrm{Hom}(A,-)(f)} & & \Big\downarrow{\scriptstyle \mathfrak{F}(f)} \\
\mathrm{Hom}(A, B') & \xrightarrow{\Phi(B')} & \mathfrak{F}(B')
\end{array}$$

commutes for any arrow $f : B \to B'$ in $\mathcal{C}$.

More generally, suppose that we have a category $\mathcal{D}$ equipped with a "forgetful" functor $\mathfrak{G} : \mathcal{D} \to \mathbf{Sets}$ then a functor $\mathfrak{F} : \mathcal{C} \to \mathcal{D}$ is $\mathfrak{G}$-representable if there exists a representation (A, Φ) such that $\mathfrak{G} \circ \mathfrak{F}$ is isomorphic to $\mathrm{Hom}(A, -)$ as functors $\mathcal{C} \to \mathbf{Sets}$.

Remark. Functors composed in natural way. If $\mathfrak{F} : \mathcal{C} \to \mathcal{C}'$ and $\mathfrak{G} : \mathcal{C}' \to \mathcal{C}''$ are co-variant functors then we have a natural composed functor $\mathfrak{G} \circ \mathfrak{F} : \mathcal{C} \to \mathcal{C}''$ where $\mathfrak{G} \circ \mathfrak{F}(A) = \mathfrak{G}(\mathfrak{F}(A))$ and $\mathfrak{G} \circ \mathfrak{F}(f) = \mathfrak{G}(\mathfrak{F}(f))$.

3.2. **Group scheme as representable functors.**

Definition 3. A group scheme over a base scheme S is a $\mathfrak{G}$-representable functor

$$\mathfrak{G} : \mathbf{Scheme}_S^{\mathrm{op}} \to \mathbf{Groups}$$

where $\mathfrak{F} : \mathbf{Groups} \to \mathbf{Sets}$ is the natural forgetful functor.

Explicitly, this simply says that a group scheme $\mathfrak{G}$ is a functor such that there exists a scheme X (called the *representing scheme*) so that for every S-scheme T, the group $\mathfrak{G}(T)$ is in bijection with S-scheme morphisms $T \to X$.

Exercise 1. Recall from [2] that any scheme can be covered by affine schemes and in fact a morphism between schemes are just a collection of "compatible" morphisms between affine schemes. Thus, if $\mathcal{C}$ is a good category (one which has product) then a functor

$$\mathfrak{G} : \mathbf{AffineScheme}_S^{\mathrm{op}} \to \mathcal{C}$$

has natural extension to a functor

$$\mathfrak{G}^+ : \mathbf{Scheme}_S^{\mathrm{op}} \to \mathcal{C}$$

by functoriality; namely, for every scheme $T \in \mathbf{Scheme}_S^{\mathrm{op}}$, we can express

$$T = \bigcup U_\alpha$$

where U_α are affine schemes. Let $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and $\iota_\alpha : U_\alpha \to T$ and $\iota_{\alpha\beta} : U_{\alpha\beta} \to U_\alpha$ natural inclusion morphisms of S-schemes. Now, should an extension $\mathfrak{G}^+$ exists, do it to the commutative diagram

$$\begin{array}{ccc} U_{\alpha\beta} & \longrightarrow & U_\alpha \\ \downarrow & & \downarrow \\ U_\beta & \longrightarrow & T \end{array}$$

we get diagram

$$\begin{array}{ccc} \mathfrak{G}(U_{\alpha\beta}) & \longleftarrow & \mathfrak{G}(U_\alpha) \\ \uparrow & & \uparrow \\ \mathfrak{G}(U_\beta) & \longleftarrow & \mathfrak{G}^+(T) \end{array}$$

by its functoriality. Thus, we should define $\mathfrak{G}^+(T) = \prod \mathfrak{G}(U_\alpha)$.

(i) Verify that $\mathfrak{G}^+$ is a well-defined functor (independent of the covering).
(ii) If $\mathfrak{G}$ is representable, is $\mathfrak{G}^+$ also representable?
(iii) Is it true that any functor $\mathfrak{G} : \mathbf{Scheme}_S^{\mathrm{op}} \to \mathbf{Groups}$ should be completely determined by its effect on the sub-category $\mathbf{AffineScheme}_S^{\mathrm{op}}$?

In particular, if $S = \mathrm{Spec}(k)$, then a group scheme functor is equivalent to a functor $\mathfrak{G} : k\text{-}\mathbf{Alg} \to \mathbf{Groups}$ that is represented by an k-algebra i.e. there exists a k-algebra R such that $\mathfrak{G}(A)$ is in one-to-one correspondence with the set of k-algebra homomorphisms $R \to A$.

Example 5. GL_n revisited. We define the functor

$$\mathfrak{G} : k\text{-}\mathbf{Alg} \to \mathbf{Groups}$$

by $\mathfrak{G}(A) = \mathsf{GL}_n(A)$. This functor is represented by the k-algebra

$$R = k[x_{ij}, y]/(\det(x_{ij})y - 1)$$

for it is easy to check that a homomorphism $\phi : R \to A$ is completely determined by the image $\phi(x_{ij}) \in A$ and conversely, for any n^2 elements a_{ij} of A such that the matrix (a_{ij}) is invertible, we get a corresponding $\phi : x_{ij} \mapsto a_{ij}; y \mapsto \frac{1}{\det(a_{ij})}$. From this example, one could realize that the functor $\mathfrak{G}$ is just a device that tells us what the "T-points" (i.e. points with coordinates in T; in algebraic geometry, one is probably familiar with the notion of k and $\overline{k}$ points of a variety but this doesn't make any sense if T is another scheme) of the underlying schemes and the "operations" on those points.

Example 6. As a special case, the multiplicative group functor $\mathbb{G}_{m/k}(A) = A^\times$ is a group scheme. It is the special case $\mathbb{G}_{m/k} = \mathsf{GL}_1$.

Example 7. The additive group functor $\mathbb{G}_{a/k}(A) = (A, +)$ is also a group scheme. What is it represented by? (Answer: The polynomial ring $R = k[x]$.)

A morphism between two group scheme functors $\mathfrak{G}$ and $\mathfrak{H}$ is just a natural transformation of functors. The equivalence of this definition and the Definition 1 is a consequence of *Yoneda lemma.*

Theorem 1 (Yoneda lemma)**.** *Let $\mathcal{C}$ be a locally small category and $\mathfrak{F} : \mathcal{C} \to$ **Sets** be a functor. Then the set of natural transformations from $\mathfrak{H}_A := Hom_{\mathcal{C}}(A, -)$ to $\mathfrak{F}$ is in bijection with $\mathfrak{F}(A)$.*

As a consequence, define a new category $\mathcal{C}^*$ whose objects are functors of the form $\mathfrak{H}_A$ for some $A \in \mathcal{C}$ and whose morphisms are natural transformations of functors. Then the category $\mathcal{C}^*$ is equivalent to $\mathcal{C}^{op}$.

Proof. Obvious.

The equivalence of category could be made explicitly in obvious way: An object $A \in \mathcal{C}$ should correspond to the functor $\mathfrak{H}_A \in \mathcal{C}^*$ and for each morphism $f : B \to A$ in $\mathcal{C}$, we have a corresponding natural transformation $\eta_f : \mathrm{Hom}_{\mathcal{C}}(A, -) \to \mathrm{Hom}_{\mathcal{C}}(B, -)$ where $\eta_f(g) = f \circ g$ for any $g \in \mathrm{Hom}_{\mathcal{C}}(A, X)$. ☐

Exercise 2 (Fun exercise)**.** Elliptic curve E over a field k is a natural a functor where for a field extension L/k, $E(L)$ is just the points projective points P with coordinates in A satisfying the defining equation of E (and group operations as in Silverman). If T is an arbitrary scheme that is not affine, how should one interpret $E(T)$ geometrically? In particular, what is $E(E)$? (This should be obvious if T is affine i.e. $T = \mathrm{Spec}(A)$ for some k-algebra A; then one can use functoriality to patch these things. As for $E(E)$, it is in bijection with $\mathrm{Hom}(E, E)$ by representability so basically, it is the group of endomorphisms of E.)

It should be clear from this functorial definition that the functor $\mathfrak{F} = \mathrm{Hom}_{\mathbf{Scheme}_S}(-, S)$ represented by S gives us the trivial group scheme i.e. $\mathfrak{F}(T)$ is trivial group for all T since there is always exactly one S-scheme homomorphism $T \to S$.

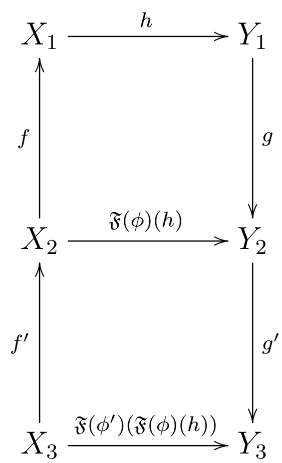
Appendix: Proof of Functoriality of Hom. *Opposite category*: Recall that for any category $\mathcal{C}$, we have the $\mathcal{C}^{\mathrm{op}}$ obtained by reversing all morphisms i.e. the objects of $\mathcal{C}^{\mathrm{op}}$ are the same as that of $\mathcal{C}$ while $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$.

Product Category: Given two categories $\mathcal{C}$ and $\mathcal{D}$, we can define a category $\mathcal{C} \times \mathcal{D}$ whose objects are pair (X, Y) of objects $X \in \mathcal{C}$, $Y \in \mathcal{D}$ and whose morphisms $(X_1, Y_1) \to (X_2, Y_2)$ are pairs of morphisms (f, g) where $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$. (Identity morphism should be obvious; as are compositions of morphisms.)

In particular, we have product $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ and if $\mathcal{C}$ is locally small, we have a canonical functor

$$\mathfrak{F} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Sets}$$

given on objects by $\mathfrak{F}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and on morphisms, say $\phi = (f, g) : (X_1, Y_1) \to (X_2, Y_2)$, by $\mathfrak{F}(\phi) : \mathrm{Hom}(X_1, Y_1) \to \mathrm{Hom}(X_2, Y_2); h \mapsto g \circ h \circ f$. Note that by definition of opposite category $f : X_2 \to X_1$ so we are just following the arrows



Is there intrinsic meaning to representable functors of $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$?

Fix an object $T \in \mathcal{C}$ then there is an obvious "embedding" functor

$$\mathfrak{E}_T : \mathcal{C} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$$

given by $\mathfrak{E}_T(X) = (T, X)$ and $\mathfrak{E}_T(f) = (\mathrm{Id}_T, f)$ for a morphism $f : X \to Y$. Composing with $\mathfrak{F}$ above, we get a functor $\mathfrak{F} \circ \mathfrak{E}_T : \mathcal{C} \to \mathbf{Sets}$ given by $X \mapsto \mathrm{Hom}(T, X)$ that we constructed at the beginning. (This shows that $\mathrm{Hom}(T, -)$ is indeed a functor.)

4. What is each definition good for?

In the theory of automorphic forms, the third definition (i.e. as representable functor) is most widely used and probably easiest to employ. That is because we get "actual groups" to analyze. It is also easy to write down the definition for the group whereas the Hopf algebra homomorphisms and the scheme morphisms are normally difficult or time consuming to describe explicitly. However, the representability condition is typically non-trivial to prove. To do that, one usually needs to construct a scheme (or a Hopf algebra) that represents the functor[4]. We shall see later that many interesting functors that are easy to write down but hard to verify to be group schemes.

Remark. We remark that there is a canonical representing scheme, at least in the affine case, in [3], example 3.7.

Suppose that a functor $\mathfrak{G} : k\text{-}\mathbf{Alg} \to \mathbf{Sets}$ is represented by an affine scheme $G = \mathrm{Spec}(A)$ i.e. $\mathfrak{G} = \mathrm{Hom}_k(A, -)$. Let $\mathbb{A}^1 : k\text{-}\mathbf{Alg} \to \mathbf{Sets}; R \mapsto R$ be the forgetful functor. Then A is isomorphic to the ring of natural transformations from $\mathfrak{G}$ to $\mathbb{A}^1$. (TODO: Add analogy with algebraic geometry. The notation $\mathbb{A}^1$ should be suggestive of the affine line.)

References

[1] Jean Dieudonné and Alexander Grothendieck. Éléments de géométrie algébrique. *Inst. Hautes Études Sci. Publ. Math.*, 4, 8, 11, 17, 20, 24, 28, 32, 1961–1967.

[2] R. Hartshorne. *Algebraic Geometry*. Encyclopaedia of mathematical sciences. Springer, 1977.

[3] James S. Milne. Basic theory of affine group schemes, 2012. Available at www.jmilne.org/math/.

[4] James S. Milne. Algebraic groups (v2.00), 2015. Available at www.jmilne.org/math/.

[4] There are criterion to test representability, as in Stack project. A good example is the relative Picard functor $Pic_{X/S}$ occurring in abelian varieties which can be described easily as group of isomorphism classes of line bundles.