CALCULUS

LAWRENCE VU

1. Inverse Functions

1.1. Review of sets and functions.

1.2. One-to-one function. $f(a) = f(b) \Rightarrow a = b$; equivalently, $a \neq b \Rightarrow f(a) \neq f(b)$

1.3. Inverse function. A function $g : B \to A$ is called *an inverse* of $f : A \to B$ if $g \circ f = \mathrm{Id}_A$ and $f \circ g = \mathrm{Id}_B$. Here, for any set A, $\mathrm{Id}_A : A \to A$ is the identity function i.e. $\mathrm{Id}_A(x) = x$ for any x in A.

1.4. Uniqueness of inverse function. If f has an inverse i.e. f is *invertible*, then there is only one inverse function so we denote f^{-1} for the unique inverse function. Fact: A function is invertible if and only if it is one-to-one.

1.5. How to find inverse function.

1.6. Graph of inverse function. Graph of f^{-1} is *reflection* of graph of f over the *diagonal line* i.e. the line $y = x$. Geometrically, it is easy to see that if f is continuous/differentiable[1] at x_0 then f^{-1} is continuous/differentiable (respectively) at $y_0 = f(x_0)$. By geometric inspection, one finds

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}.$$

2. Logarithm and Exponential

2.1. Definition. The natural logarithm function $\ln : (0, +\infty) \to \mathbb{R}$ is given by

$$\ln(x) := \int_1^x \frac{1}{t} dt.$$

Exercise: Think of what happen if you compute the integral for $x < 0$.

2.2. Algebraic properties of logarithm. ln converts multiplication to addition:

(1) $\ln(xy) = \ln(x) + \ln(y)$
(2) $\ln(\frac{x}{y}) = \ln(x) - \ln(y)$
(3) $\ln(x^r) = r\ln(x)$

Prove the first with Mean Value Theorem; the remaining rules are consequences.

[1] Derivative exists; or equivalently, tangent line exists.

2.3. **Analytic properties of logarithm.** By Fundamental Theorem of Calculus, $\ln(x)$ is differentiable at any $a \in (0, +\infty)$ and

$$\ln'(a) = \frac{1}{a}.$$

This implies that

(1) ln is increasing (by MVT, hence, is one-to-one),
(2) ln is also continuous,
(3) $\lim_{x\to+\infty} \ln(x) = +\infty$ and $\lim_{x\to 0^+} \ln(x) = -\infty$,
(4) ln achieves all values in $\mathbb{R}$ (by IVT).

2.4. **Applications of logarithm.** Logarithmic derivatives. Indefinite integral of $\frac{1}{x}$ and $\tan(x)$:

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int \tan(x) dx = -\ln|\cos(x)| + C$$

2.5. **Exponential function.** Since $\ln(x)$ is one-to-one, it is invertible and its (unique) inverse function we denote $\exp(x) : \mathbb{R} \to (0, +\infty)$. By definition,

$$\exp(\ln(x)) = x \qquad \text{for all positive real numbers } x > 0$$

and

$$\ln(\exp(x)) = x \qquad \text{for all real number } x.$$

We also write e^x for $\exp(x)$.

2.6. **Algebraic properties of exponential.** exp function converts addition to multiplication:

(1) $e^{x+y} = e^x e^y$
(2) $e^{x-y} = \frac{e^x}{e^y}$
(3) $(e^x)^r = e^{rx}$

2.7. **Analytic properties of exponential.** By general properties of inverse function:

(1) e^x is continuous and differentiable at all point in the domain [since its graph is the reflection of the graph of ln].
(2) $\exp(x)$ is its own derivative/indefinite integral i.e. $\exp'(a) = \exp(a)$ for any $a \in \mathbb{R}$; and so $\int e^x = e^x + C$.
(3) $\exp(x)$ is increasing.

2.8. **General logarithm and exponentials.** For any fixed positive real $a > 0$ and $a \neq 1$, we define the exponential function with base a,

$$a^x : \mathbb{R} \to (0, +\infty) \qquad\qquad a^x = e^{x\ln(a)}.$$

This function satisfies similar properties to the natural exponential e^x. In particular, it is one-to-one whose inverse we denote by $\log_a(x)$. It is easy to see that[2]

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}.$$

Exercise: Derive the formulas for derivatives and indefinite integrals by yourself.

[2]This is normally referred to as the "change of base" formula.

2.9. **Applications.** Let $c \in \mathbb{R}$ be a fixed constant. The general solution to the differential equation $\frac{dP}{dt} = cP$ is given by $P(t) = \alpha e^{ct}$ where $\alpha = P(0)$ is the initial value.

Many natural phenomenon such as population and radioactive decay can be modelled by the above differential equation.

3. Inverse Trigonometric and Hyperbolic Functions

3.1. **Inverse trigonometric functions.** The function $\sin(x) : \mathbb{R} \to [-1, 1]$ is not one-to-one but its restriction $\sin(x) : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ is and hence has the inverse function

$$\sin^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

which we also denoted by arcsin.

As typical with inverses, to compute $\sin^{-1}(t)$, you look for the angle α such that $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ such that $\sin(\alpha) = t$ [by the circle method]. For example, $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$.

Likewise, we have other functions

$$\cos^{-1} : [-1, 1] \to [0, \pi] \qquad\qquad \tan^{-1} : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Pay close attention to the domain and range of these functions!
Exercise: Find derivative of these functions and describe their analytic properties.
Exercise: Describe $\tan^{-1}$ in term of $\sin^{-1}$ and compute derivative of $\tan^{-1}$ using derivative of $\sin^{-1}$.

3.2. **Hyperbolic functions.**

$$\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad :\mathbb{R} \to \mathbb{R}$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad :\mathbb{R} \to [1, +\infty)$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad :\mathbb{R} \to (-1, 1)$$

There inverses can be computed explicitly in term of natural logarithm

$$\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1}) \qquad :\mathbb{R} \to \mathbb{R}$$

$$\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1}) \qquad :[1, +\infty) \to [0, +\infty)$$

$$\tanh^{-1}(x) = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right) \qquad :(-1, 1) \to \mathbb{R}$$

Exercise: Use the general method of computing inverse to derive the above formulas. (Write down the equation $y = \sinh(x)$; swap two variables to get equation $x = \sinh(y)$ which is equivalent to $\frac{e^x - e^{-x}}{2} = y$; and solve for y in term of x.)
Exercise: Find derivative of these functions.

4. L'Hospital's rule

Suppose f, g are differentiable at a and $g'(x) \neq 0$ near a i.e. in $(a - \epsilon, a) \cup (a, a + \epsilon)$ for some $\epsilon > 0$ small enough. If

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = \pm\infty$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right hand side exists (or $\pm\infty$).

We can replace "$x \to a$" by "$x \to a^+$", "$x \to a^-$", "$x \to +\infty$" and "$x \to -\infty$".

5. Integration by Parts

If $u, v : I \to \mathbb{R}$ then

$$\int u\ dv = uv - \int v\ du$$

Explicitly,

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx$$

For definite integral:

$$\int_a^b u(x)v'(x)dx = u(x)v(x)]_a^b - \int_a^b v(x)u'(x)dx$$

Like many other problems in math such as solving equations, integration by part normally requires practice and experience to master.

Exercise: Compute $\int x^n e^x\ dx$ for $n = 1, 2, 3, 4$.

Exercise: Compute $\int \sin(x)e^x\ dx$.

Exercise: Compute $\int \ln^n(x)\ dx$ for $n = 1, 2, 3, 4$.

6. Integration of Trigonometric Functions

The trick to compute integrals of the form $P(\sin x, \cos x)$ where $P(Z, T)$ is a rational function in Z, T; for example,

$$\int \cos^3(x)dx \qquad \text{for } P(Z,T) = T^3$$

$$\int \sin^5(x)\cos^2(x)dx \qquad \text{for } P(Z,T) = Z^5T^2$$

$$\int \tan^3(x)dx \qquad \text{for } P(Z,T) = \frac{Z^3}{T^3}$$

is to manipulate the integral to reduce the problem to computing integrals of a simple trigonometric function or integrals of polynomial/rational function.

For example, one can rewrite the first integrand

$$\begin{aligned}\cos^3(x)\,dx &= \cos(x)(1-\sin^2(x))\,dx && \text{from the identity } \sin^2(x)+\cos^2(x)=1\\ &= (1-\sin^2(x))\,d\sin(x)\end{aligned}$$

and likewise for the second integrand

$$\begin{aligned}\sin^5(x)\cos^2(x)\,dx &= \sin(x)\sin^4(x)\cos^2(x)\,dx\\ &= \sin(x)(1-\cos^2(x))^2\cos^2(x)\,dx && \text{from the identity } \sin^2(x)+\cos^2(x)=1\\ &= -(1-\cos^2(x))^2\cos^2(x)\,d\cos(x)\end{aligned}$$

and then use substitution rule.

6.1. **Inverse substitution.** If you see the expressions x^2+a^2, a^2-x^2 or x^2-a^2 appearing under square root, it sometimes help to perform inverse substitution $x = a\tan\theta$, $x = a\sin\theta$ or $x = a\sec\theta$ respectively. For then

$$\begin{aligned} x^2+a^2 &= (a\tan\theta)^2+a^2 = a^2(\tan^2\theta+1) = a^2\sec^2\theta \\ a^2-x^2 &= a^2-(a\sin\theta)^2 = a^2(1-\sin^2\theta) = a^2\cos^2\theta \\ x^2-a^2 &= (a\sec\theta)^2-a^2 = a^2(1-\sec^2\theta) = a^2\tan^2\theta \end{aligned}$$

become perfect square and so cancel the outer square root.
Example: The area of an ellipse of radii a and b can be computed using

$$4\int_0^a \frac{b}{a}\sqrt{a^2-x^2}dx$$

Perform substitution $x = a\sin\theta$ with $\theta \in [0, \frac{\pi}{2}]$ allows you to compute the integral.

7. Integration of Rational Functions

For rational functions $\frac{P(x)}{Q(x)}$ where P, Q are polynomials in x, we use the method of *partial fractions*. The idea is to rewrite it as sum of a polynomial and simple fractions of the form $\frac{A}{(ax+b)^i}$ or $\frac{Ax+B}{(ax^2+bx+c)^i}$ each of which can be easily integrated. For example, if one recognizes the identity

$$\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

he/she can easily integrate the left hand side, namely

$$\int \frac{1}{x^2-1} = \frac{1}{2}(\ln|x-1| - \ln|x+1|) + C.$$

Here are the step to express a rational function as partial fractions:

Step 1 If the degree of P is greater than the degree of Q, do long division to write

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where $S(x)$ and $R(x)$ are polynomials and degree of R is less than that of Q.

Step 2 Factorize $Q(x)$ into linear and quadratic factors of negative discriminant i.e.

$$Q(x) = A\prod_{i=1}^{k}(x+a_i)^{r_i}\prod_{i=1}^{m}(x^2+b_ix+c_i)^{s_i}$$

where A, a_i, b_i, c_i are real numbers; $a_i \neq a_j$ for $i \neq j$ and likewise for b_i, c_i; also $b_i^2 - 4c_i < 0$. Note that k or m can be zero i.e. there is no linear or quadratic factor appearing in $Q(x)$.

Note: The book allows for more general linear factor a_ix+b_i but this makes it hard for you to recognize if two factors are the same. For example, $3x-1$ and $x-\frac{1}{3}$ are the "same" and so $(3x-1)(x-\frac{1}{3})$ should be treated as $3(x-\frac{1}{3})^2$.

Step 3 Write down the form of partial fraction

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^{k}\left(\sum_{j=1}^{r_i}\frac{A_{i,j}}{(x+a_i)^j}\right) + \sum_{i=1}^{m}\left(\sum_{j=1}^{s_i}\frac{B_{i,j}x+C_{i,j}}{(x^2+b_ix+c_i)^j}\right)$$

where $A_{i,j}, B_{i,j}, C_{i,j}$ are "new" symbols representing real numbers.

Step 4 Solve for the $A_{i,j}, B_{i,j}, C_{i,j}$ so that the identity above holds.

Note: Normally, you will get a huge system of linear equations. But substituting appropriate value of x might solve the problem immediately.

Step 5 Compute integral.

Example: Let us compute

$$\int \frac{4x-1}{x^2(x-1)(x^2+1)} dx$$

In this case, we don't have to perform Step 1 (as the numerator $4x-1$ is of degree 1 and the denominator is of greater degree 5) or Step 2 since the denominator is already in factored form (we have $k=2$ distinct linear factor (x and $x-1$) and $m=1$ quadratic factor (x^2+1). For step 3, the partial fraction form should be

$$\frac{4x-1}{x^2(x-1)(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+1}$$

for some real number A, B, C, D. Clear denominators on both sides one has identity

$$4x-1 = Ax(x-1)(x^2+1) + B(x-1)(x^2+1) + Cx^2(x^2+1) + (Dx+E)x^2(x-1)$$

and multiplying everything out:

$$4x-1 = (A+C+D)x^4 + (B-A-D+E)x^3 + (A-B+C-E)x^2 + (B-A)x - B$$

and so to match both sides, we need

$$\begin{cases} A+C+D=0 \\ B-A-D+E=0 \\ A-B+C-E=0 \\ B-A=4 \\ -B=-1 \end{cases}$$

The last condition implies $B=1$. Then the one before it yields $A = B-4 = 1-4 = -3$. Adding the second and third gives $C-D=0$ so $D=C$. So the first gives $A+2C=0$ and thus $C = \frac{-A}{2} = \frac{3}{2}$. Then from the third, we easily get $E = A-B+C = -3-1+\frac{3}{2} = -1-\frac{3}{2} = -\frac{5}{2}$. Hence, we find

$$\begin{aligned} \int \frac{4x-1}{x^2(x-1)(x^2+1)} dx &= \int \left(\frac{-3}{x} + \frac{1}{x^2} + \frac{3}{2(x-1)} + \frac{3x-5}{2(x^2+1)} \right) dx \\ &= -3\ln|x| - \frac{1}{x} + \frac{3}{2}\ln|x-1| + \frac{3}{4}\int \frac{2x}{x^2+1} dx - \frac{5}{2}\int \frac{1}{x^2+1} dx \\ &= -3\ln|x| - \frac{1}{x} + \frac{3}{2}\ln|x-1| + \frac{3}{4}\ln|x^2+1| - \frac{5}{2}\tan^{-1}(x) + C_0 \end{aligned}$$

Exercise: Compute the integral

$$\int \frac{A}{(ax+b)^n} \, dx$$

where $A, a, b \in \mathbb{R}$ are fixed and $a \neq 0$.

Exercise: Compute the integral

$$\int \frac{Ax+B}{(ax^2+bx+c)^n} \, dx$$

where $A, B, a, b, c \in \mathbb{R}$ are fixed, $a \neq 0$ and $b^2 - 4ac < 0$. Hint: Try splitting the integrand $\frac{Ax+B}{(ax^2+bx+c)^n} = \frac{Ax+C}{(ax^2+bx+c)^n} + \frac{B-C}{(ax^2+bx+c)^n}$ for some appropriate C so that the integral $\int \frac{Ax+C}{(ax^2+bx+c)^n} \, dx$ could be solved with substitution $u = ax^2+bx+c$. To compute $\int \frac{B-C}{(ax^2+bx+c)^n} \, dx$, complete the square in the denominator and perform trigonometric substitution.

8. Length of cardioid

Cardioid equation $r(\theta) = 1 + \sin\theta$. The length of curve given by polar coordinate equation is given by

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{r(\theta)^2 + r'(\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{1 + \sin^2\theta + 2\sin\theta + \cos^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta \\ &= \sqrt{2}\int_0^{2\pi} \sqrt{1+\sin\theta}\, d\theta \end{aligned}$$

To compute the last integral, we need a remove the square root somehow. Observe that should it be $1 + \cos\theta$, we can use the half formula

$$\cos\theta = 2\cos^2\left(\frac{\theta}{2}\right) - 1$$

to derive

$$1 + \cos\theta = 2\cos^2\left(\frac{\theta}{2}\right)$$

and so taking square-root reduces to a simple integral. To turn $\sin(\theta)$ to $\cos(\theta)$, we observe the identity

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x.$$

And to exploit that identity, we make substitution $u = \frac{\pi}{2} - \theta \Rightarrow \theta = \frac{\pi}{2} - u, d\theta = -du$ to continue

$$\begin{aligned} L &= \sqrt{2}\int_0^{2\pi} \sqrt{1+\sin\theta}\, d\theta \\ &= \sqrt{2}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}-2\pi} \sqrt{1+\sin\left(\frac{\pi}{2} - u\right)}\, (-du) \\ &= \sqrt{2}\int_{-\frac{3\pi}{2}}^{\frac{\pi}{2}} \sqrt{1+\cos(u)}\, du \\ &= \sqrt{2}\int_{-\frac{3\pi}{2}}^{\frac{\pi}{2}} \sqrt{2\cos^2\left(\frac{u}{2}\right)}\, du && \text{by the identity} \\ &= 2\int_{-\frac{3\pi}{2}}^{\frac{\pi}{2}} \left|\cos\left(\frac{u}{2}\right)\right|\, du \end{aligned}$$

Now substitute $v = \frac{u}{2}$ then $u = 2v$ and $du = 2dv$. We have

$$\begin{aligned}
L &= 2\int_{-\frac{3\pi}{2}}^{\frac{\pi}{2}} \left|\cos\left(\frac{u}{2}\right)\right| \, du \\
&= 2\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} |\cos(v)| \, 2dv \\
&= 4\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} |\cos(v)| \, dv \\
&= 4\left[\int_{-\frac{3\pi}{4}}^{-\frac{\pi}{2}} (-\cos(v)) \, dv + \int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos(v) \, dv\right]
\end{aligned}$$

since $\cos(v) < 0$ for $-\frac{3\pi}{4} \le v < -\frac{\pi}{2}$ and $\cos(v) \ge 0$ for $-\frac{\pi}{2} \le v \le \frac{\pi}{4}$. We can now resolve

$$\begin{aligned}
L &= 4\left[-\sin(v)|_{-\frac{3\pi}{4}}^{-\frac{\pi}{2}} + \sin(v)|_{-\frac{\pi}{2}}^{\frac{\pi}{4}}\right] \\
&= 4\left[-\underbrace{\sin(-\frac{\pi}{2})}_{-1} \underbrace{+\sin(-\frac{3\pi}{4}) + \sin(\frac{\pi}{4})}_{0} - \sin(-\frac{\pi}{2})\right] \\
&= 8
\end{aligned}$$

9. Legnth of the 4-leaf curve

The 4-leaf curve is given by $r = \cos(2\theta)$. So its length is given by

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\cos^2(2\theta) + (-2\sin(2\theta))^2} \, d\theta \\
&= \int_0^{2\pi} \sqrt{\cos^2(2\theta) + 4\sin^2(2\theta)} \, d\theta \\
&= \int_0^{2\pi} \sqrt{1 + 3\sin^2(2\theta)} \, d\theta \\
&\approx 9.68845...
\end{aligned}$$

(This is an example of an elliptic integral and it doesn't seem to have easy form.)