Calculus II - Exam 3

7 May 2018

Problems

1. Find the area of the region bounded by the two curves

$$x = y^2 - 6 \quad \text{and} \quad x = e^y$$

and the two lines

$$y = 1 \quad \text{and} \quad y = -1.$$

2. Find the volume V of the solid obtained by rotating the region bounded by the curves

$$y = 5x \quad \text{and} \quad y = 5\sqrt{x}$$

about the line $y = 5$.

3. Use the method of cylindrical shells to find the volume V generated by rotating the region bounded by the following curves and lines

$$y = 11e^{-x^2}, \quad y = 0, \quad x = 0, \quad \text{and } x = 1$$

about the y-axis.

4. Find the exact length of the curve

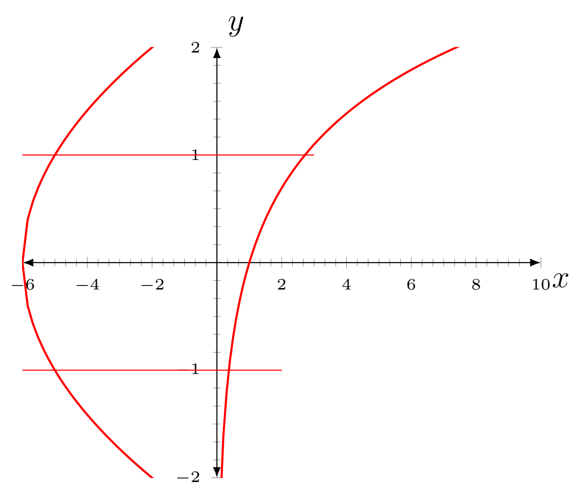
$$y = \ln(1 - x^2), \quad 0 \leq x \leq \frac{1}{7}$$

5. A leaky 4 lb bucket and a rope of negligible weight are used to draw water from a well that is 100 ft deep. The bucket is filled with 50 lb of water and pulled up at a rate of 2 ft/s. The water leaks at the constant rate and when the bucket reaches the top, the bucket has 30 lb of water left. How much work is done in puling this bucket to the top of the well?

Solution

1. From the graph, the area is given by

$$\begin{aligned} A &= \int_{-1}^{1} (e^y - (y^2 - 6))\, dy \\ &= e^y - \frac{y^3}{3} + 6y \Big|_{-1}^{1} \\ &= e - \frac{1}{e} - \frac{2}{3} + 12 \\ &= e - \frac{1}{e} + \frac{34}{3} \end{aligned}$$



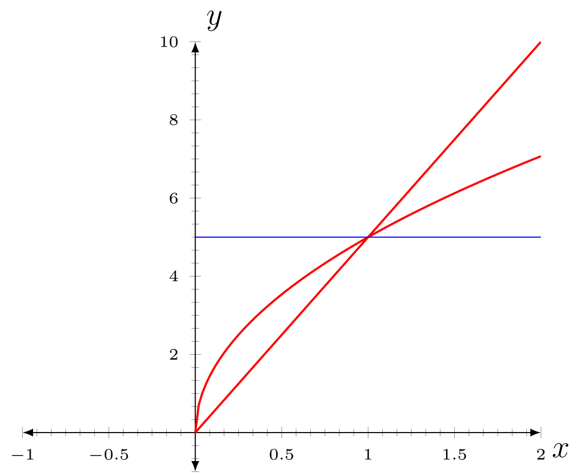
If you can't draw the graph, you can also analyze as follow: Recall that the area is always given by

$$A = \int_{-1}^{1} |y^2 - 6 - e^y|\, dy$$

The main issue is to remove the absolute value. To do that, we need to identify the intervals in which $y^2 - 6 - e^y \geq 0$.

Let $f(y) = y^2 - 6 - e^y$. Then $f'(y) = 2y - e^y$ and $f''(y) = 2 - e^y < 0$ if $y > \ln(2)$ so the function $f'(y)$ is decreasing on $(\ln(2), +\infty)$ and increasing on $(-\infty, \ln(2))$. Thus, the function $f'(y)$ achieves maximum value at $y = \ln(2)$ and the maximum value is $2\ln(2) - e^{\ln(2)} = 2\ln(2) - 2 < 0$ which is a negative number since $\ln(2) < 1$. This shows that $f'(y) < 0$ for all y. Hence, f is decreasing and this shows that $f(y) \leq f(-1) = -5 - e^{-1} < 0$ for all $y \in [-1, 1]$. Thus, $|f(y)| = -f(y)$ for all y.

2. The intersections of the two curves are solution to $5x = 5\sqrt{x} \iff x^2 = x \iff x = 0, x = 1$.



Observe that a cross-section of the solid is bounded by two concentric circles of outer radius $5 - 5x$ and inner radius $5 - 5\sqrt{x}$; hence, the area of a cross-section at x is

$$\begin{aligned}
\pi(5-5x)^2 - \pi(5-5\sqrt{x})^2 &= \pi(5-5x-(5-5\sqrt{x}))(5-5x+(5-5\sqrt{x})) \\
&= \pi(5\sqrt{x}-5x)(10-5x-5\sqrt{x}) \\
&= 25\pi(\sqrt{x}-x)(2-x-\sqrt{x}) \\
&= 25\pi(2\sqrt{x}-2x-(\sqrt{x}-x)(x+\sqrt{x})) \\
&= 25\pi(2\sqrt{x}-2x-(x-x^2)) \\
&= 25\pi(x^2+2\sqrt{x}-3x)
\end{aligned}$$

The volume is therefore

$$\int_0^1 25\pi(x^2+2\sqrt{x}-3x)dx = 25\pi\left(\frac{x^3}{3}+\frac{4}{3}x^{3/2}-\frac{3x^2}{2}\right)\Big|_0^1 = 25\pi\left(\frac{1}{3}+\frac{4}{3}-\frac{3}{2}\right) = \frac{25\pi}{6}$$

3. Using the formula given in class for the cylindrical shell method

$$V = \int_a^b 2\pi x f(x)dx,$$

so in this case with $f(x) = 11e^{-x^2}$, we find that the volume is

$$\int_0^1 2\pi x(11e^{-x^2})dx = -11\pi\int_0^1 e^{-x^2}d(-x^2) = -11\pi e^{-x^2}\Big|_0^1 = 11\pi\left(1-\frac{1}{e}\right).$$

4. Recall the length formula derived in class

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$$

so in this case, with $y = \ln(1-x^2)$, $a = 0$, $b = 1/7$, we get $\frac{dy}{dx} = \frac{-2x}{1-x^2}$ and so the length of the curve is

$$\int_0^{1/7} \sqrt{1 + \left(\frac{-2x}{1-x^2}\right)^2} dx$$
$$= \int_0^{1/7} \sqrt{1 + \frac{4x^2}{(1-x^2)^2}} dx$$
$$= \int_0^{1/7} \sqrt{\frac{(1-x^2)^2 + 4x^2}{(1-x^2)^2}} dx$$
$$= \int_0^{1/7} \sqrt{\frac{(1+x^2)^2}{(1-x^2)^2}} dx$$
$$= \int_0^{1/7} \frac{1+x^2}{1-x^2} dx \qquad \text{since } 1+x^2, 1-x^2 \geq 0 \text{ for all } 0 \leq x \leq 1/7$$
$$= \int_0^{1/7} \left(-1 + \frac{2}{1-x^2}\right) dx$$
$$= \int_0^{1/7} \left(-1 + \frac{1}{1-x} + \frac{1}{1+x}\right) dx \qquad \text{by method of partial fractions}$$
$$= \ln|x+1| - \ln|x-1| - x \quad \Big]_0^{1/7}$$
$$= \ln\frac{8}{7} - \ln\frac{6}{7} - \frac{1}{7}$$
$$= \ln\frac{4}{3} - \frac{1}{7}$$

5. Make the coordinate system going upward from the bottom of the well. To compute the work, we need to find the force needed to pull the bucket at the distant x ft ($0 \leq x \leq 100$). By assumption, the total amount of leakage is $50 - 30 = 20$ lb of water so the amount at x feet, the amount of leakage should be $20\frac{x}{100}$ i.e. we have $50 - 20\frac{x}{100}$ lb of water at height x ft. The total weight of the bucket and water at height x is thus $4 + 50 - 20\frac{x}{100}$. This is the total amount of force we need to exert to overcome gravity; in other words, the force function is

$$f(x) = 4 + 50 - 20\frac{x}{100} = 54 - \frac{x}{5}$$

It follows that the work is

$$\int_0^{100} f(x)dx = \int_0^{100} \left(54 - \frac{x}{5}\right) dx = 54x - \frac{x^2}{10}\Big|_0^{100} = 5400 - \frac{100^2}{10} = 5400 - 1000 = 4400$$