Calculus II - Exam 2

28 March 2018

Problems

1. State the integration by parts formula. Then compute the following integrals

   (a) $\int (x^2 + 8x)\cos(x)\, dx$ (b) $\int \sin^{-1}(x)\, dx$ (c) $\int \frac{\ln(x)}{\sqrt{x}} dx$

2. (a) Use integration by parts technique, obtain reduction formula for $\int \cos^n(x)dx$ where $n \geq 2$ is a natural number. In other words, express $\int \cos^n(x)dx$ in term of the simpler integral $\int \cos^{n-2}(x)dx$.

   (b) Use your formula in part (a) to compute $\int \cos^3(x)dx$.

   (c) In class, I mentioned that an alternative way to compute the above integral was to use trigonometric identity. Use the identity $\cos(3x) = \cos^3(x) - 3\sin^2(x)\cos(x)$ to re-compute the integral in part (b). Compare the answer you get with what you get in part (b); you will find the identity $\sin(3x) = 3\sin(x)\cos^2(x) - \sin^3(x)$ helpful.

3. Evaluate

   (a) $\int \sin^3(x)\cos^3(x)dx$ (b) $\int 9\sin(x)\cos^4(x)dx$ (c) $\int 2\tan(x)\sec^3(x)dx$

4. State the inverse substitution formula: *If* $g : [\alpha, \beta] \to [a, b]$ *is one-to-one continuous function then* $\int_a^b f(x)dx =$? Then evaluate

   (a) $\int \frac{x^3}{\sqrt{x^2+36}} dx$ (b) $\int \frac{1}{x^3\sqrt{x^2-9}} dx$ (c) $\int 9x\sqrt{1-x^4}dx$

5. Using the method of partial fractions, evaluate

   (a) $\int \frac{x-8}{x^2-7x+10} dx$ (b) $\int \frac{x^3-2x+11}{x^2-x-6} dx$ (c) $\int \frac{1}{(x-1)(x^2+2x+5)} dx$

Solution

1. State the integration by parts formula:

$$\int f dg = fg - \int g df$$

(a)

$$\begin{aligned}
\int (x^2+8x)\cos(x)\,dx &= \int (x^2+8x)d(\sin(x)) \\
&= (x^2+8x)\sin(x) - \int \sin(x)(2x+8)dx && \text{i. b. p. with } f = x^2+8x, g=\sin(x) \\
&= (x^2+8x)\sin(x) + 2\int x(-\sin(x))dx - 8\int \sin(x)dx \\
&= (x^2+8x)\sin(x) + 2\int xd(\cos x) + 8\cos(x) \\
&= (x^2+8x)\sin(x) + 2\left(x\cos x - \int \cos(x)dx\right) + 8\cos(x) && \text{i. b. p. with } f = x, g = \cos(x) \\
&= (x^2+8x)\sin(x) + 2x\cos x - 2\sin(x) + 8\cos(x)
\end{aligned}$$

(b)

$$\begin{aligned}
\int \sin^{-1}(x)\,dx &= \sin^{-1}(x)x - \int xd(\sin^{-1}(x)) && \text{i. b. p. with } f = \sin^{-1}(x), g = x \\
&= \sin^{-1}(x)x - \int x\frac{1}{\sqrt{1-x^2}}dx && \text{recall that } (\sin^{-1}(x))' = \frac{1}{\sqrt{1-x^2}} \\
&= x\sin^{-1}(x) + \sqrt{1-x^2}
\end{aligned}$$

To compute the remaining integral, do the substitution $u = 1 - x^2$ so $du = -2xdx$ and so

$$\begin{aligned}
\int x\frac{1}{\sqrt{1-x^2}}dx &= \int \frac{xdx}{\sqrt{1-x^2}} \\
&= \int \frac{-1/2du}{\sqrt{u}} \\
&= -\frac{1}{2}\int u^{-1/2}du \\
&= -\frac{1}{2}\frac{u^{1/2}}{1/2} \\
&= -\sqrt{u} \\
&= -\sqrt{1-x^2}
\end{aligned}$$

2. (a)

$$\begin{aligned}
\int \cos^n(x)dx &= \int \cos^{n-1}(x)\cos(x)dx \\
&= \int \cos^{n-1}(x)d(\sin(x)) \\
&= \cos^{n-1}(x)\sin(x) - \int \sin(x)d(\cos^{n-1}(x)) && \text{i. b. p.} \\
&= \cos^{n-1}(x)\sin(x) - \int \sin(x)(n-1)\cos^{n-2}(x)(-\sin(x))dx \\
&= \cos^{n-1}(x)\sin(x) + (n-1)\int \sin^2(x)\cos^{n-2}(x)dx \\
&= \cos^{n-1}(x)\sin(x) + (n-1)\int (1-\cos^2(x))\cos^{n-2}(x)dx \\
&= \cos^{n-1}(x)\sin(x) + (n-1)\int (\cos^{n-2}(x) - \cos^n(x))dx \\
&= \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)dx - (n-1)\int \cos^n(x)dx
\end{aligned}$$

From the equation

$$\int \cos^n(x)dx = \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)dx - (n-1)\int \cos^n(x)dx$$

we get

$$n\int \cos^n(x)dx = \cos^{n-1}(x)\sin(x) + (n-1)\int \cos^{n-2}(x)dx$$

by adding $(n-1)\int \cos^n(x)dx$ to both sides. Dividing both sides by n, we obtain the reduction formula

$$\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx$$

(b) By the above formula for $n = 3$, we have

$$\begin{aligned}\int \cos^3(x)dx &= \frac{1}{3}\cos^2(x)\sin(x) + \frac{2}{3}\int \cos(x)dx \\ &= \frac{1}{3}\cos^2(x)\sin(x) + \frac{2}{3}\sin(x)\end{aligned}$$

3. (a)

$$\begin{aligned}\int \sin^3(x)\cos^3(x)dx &= \int \sin^3(x)\cos^2(x)d(\sin(x)) \\ &= \int \sin^3(x)(1-\sin^2(x))d(\sin(x)) \\ &= \int (\sin^3(x) - \sin^5(x))d(\sin(x)) \\ &= \frac{\sin^4(x)}{4} - \frac{\sin^6(x)}{6}\end{aligned}$$

(b)

$$\begin{aligned}\int 9\sin(x)\cos^4(x)dx &= -9\int \cos^4(x)d(\cos(x)) \\ &= -9\frac{\cos^5(x)}{5}\end{aligned}$$

4. State the inverse substitution formula: *If $g : [\alpha, \beta] \to [a, b]$ is one-to-one continuous function then*

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t)dt$$

(a) Substitute $x = 6\tan(\theta)$

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+36}}dx &= \int \frac{(6\tan(\theta))^3}{\sqrt{(6\tan(\theta))^2+36}}6\sec^2(\theta)d\theta \\ &= \int \frac{(6\tan(\theta))^3}{6\sec(\theta)}6\sec^2(\theta)d\theta \\ &= 6^3\int \tan^3(\theta)\sec(\theta)d\theta \\ &= 6^3\int \tan^2(\theta)d(\sec\theta) \\ &= 6^3\int (\sec^2(\theta)-1)d(\sec\theta) \\ &= 6^3\left(\frac{\sec^3(\theta)}{3} - \sec(\theta)\right) \\ &= 6^3\left(\frac{(\sqrt{(x/6)^2+1})^3}{3} - \sqrt{(\frac{x}{6})^2+1}\right) \\ &= 6^3\left(\frac{1/6^3(x^2+36)^{3/2}}{3} - \frac{1}{6}\sqrt{x^2+36}\right) \\ &= \frac{(x^2+36)^{3/2}}{3} - 36\sqrt{x^2+36}\end{aligned}$$

Alternatively, you can do substitution $u = x^2 + 36$.

(b) Substitute $x = 3\sec\theta$ so that $\sqrt{x^2-9} = \sqrt{9\sec^2(\theta)-9} = 3\sqrt{\sec^2(\theta)-1} = 3\tan(\theta)$ and so

$$\begin{aligned}
\int \frac{1}{x^3\sqrt{x^2-9}}dx &= \int \frac{1}{3^3\sec^3(\theta)3\tan\theta}3\tan\theta\sec\theta d\theta \\
&= \frac{1}{27}\int \frac{1}{\sec^2(\theta)}d\theta \\
&= \frac{1}{27}\int \cos^2(\theta)d\theta \\
&= \frac{1}{27}\left(\frac{1}{2}\cos(\theta)\sin(\theta) + \frac{1}{2}\theta\right) && \text{by question 2(a)} \\
&= \frac{1}{54}\left(\cos(\theta)\sin(\theta) + \theta\right) \\
&= \frac{1}{54}\left(\frac{3}{x}\frac{\sqrt{x^2-9}}{x} + \sec^{-1}\left(\frac{x}{3}\right)\right)
\end{aligned}$$

5. (a) Note that $x^2 - 7x + 10 = (x-2)(x-5)$. So we express

$$\frac{x-8}{x^2-7x+10} = \frac{A}{x-2} + \frac{B}{x-5}$$

for suitable A, B. Clearing denominator, we get

$$x - 8 = A(x-5) + B(x-2)$$

Plugging in $x = 5$ yields $B = -1$ and then plugging in $x = 2$ yields $A = 2$. Hence,

$$\begin{aligned}
\int \frac{x-8}{x^2-7x+10}dx &= \int \left(\frac{2}{x-2} - \frac{1}{x-5}\right)dx \\
&= 2\ln|x-2| - \ln|x-5|
\end{aligned}$$

(b) Since the numerator has higher degree than the denominator, we first do long division

$$\frac{x^3-2x+11}{x^2-x-6} = x + 1 + \frac{5x+17}{x^2-x-6}$$

and then express the remaining part as partial fractions

$$\frac{5x+17}{x^2-x-6} = \frac{32}{5(x-3)} - \frac{7}{5(x+2)}$$

Thus

$$\begin{aligned}
\int \frac{x^3-2x+11}{x^2-x-6}dx &= \int \left(x + 1 + \frac{32}{5(x-3)} - \frac{7}{5(x+2)}\right)dx \\
&= \frac{x^2}{2} + x + \frac{32}{5}\ln|x-3| - \frac{7}{5}\ln|x+2|
\end{aligned}$$