Calculus II - Exam 1

21 February 2018

Problems

1. (a) State the definition of one-to-one function.
   (b) Check if the function given by the expression $f(x) = 3 + \sqrt{4 + 5x}$ on its largest domain is one-to-one by the definition.
   (c) If the function is one-to-one, find its inverse and determine the domain of the inverse function.

2. (a) Write down the formula for the derivative of the inverse function.
   (b) The function $f : \left[\frac{1}{2}, \frac{3}{2}\right] \to \mathbb{R}$ given by
   $$f(x) = \ln\left(\frac{x+1}{1-\sin(x)}\right)$$
   is one-to-one (in fact, increasing). Find the derivative
   $$(f^{-1})'(a)$$
   where
   $$a = f(1) = \ln\left(\frac{2}{1-\sin(1)}\right).$$
   (c) Write the equation of the tangent line to the graph of f^{-1} at the point $(a, f^{-1}(a))$.

3. Evaluate the integral
   $$\int_1^e \frac{3x^2+2x+1}{x}dx$$

4. (a) Express $\tan^{-1}(x)$ as a composition of $\sin^{-1}(x)$ and some algebraic function of x. (An algebraic function here means a function that only involves addition, multiplication, fractions and taking roots such as $\frac{x}{\sqrt{x-1}}$ or $\frac{x\sqrt[3]{x}}{1+x+x^2}$.)
   (b) Compute derivative of $\tan^{-1}$ using the expression obtained in part (a) and chain rule.
   (c) Compute the derivative of $\tan^{-1}$ as inverse function of tan (i.e. via the formula of question 2, part (a)). Verify that the result agrees with your answer to part (b).

5. (a) State L'Hospital's rule.
   (b) Compute the limit
   $$\lim_{x\to 0^+} \tan(8x^2)^x$$

Solution

1. (a) A function f is one-to-one if for any a, b in the domain of f, if $a \neq b$ then $f(a) \neq f(b)$; or equivalently, if $f(a) = f(b)$ then $a = b$.

   (b) By definition, we need to check that for any real number a, b in the domain of f, if $f(a) = f(b)$ then $a = b$. Now, $f(a) = f(b)$ means $3 + \sqrt{4+5a} = 3 + \sqrt{4+5b}$ by definition of the function f which implies $\sqrt{4+5a} = \sqrt{4+5b}$ so $4 + 5a = 4 + 5b$ and thus $a = b$.

   Remark: Some of you showed that the function is increasing so it is one-to-one. This is technically incorrect because this is not what the question asks for (you are supposed to illustrate that the function is one-to-one BY THE DEFINITION); but I am lenient here. Any other kind of answer (e.g. using graph or checking several values) is deemed incorrect.

   (c) Set $y = 3 + \sqrt{4+5x}$ and solve for x:

$$\begin{aligned} y = 3 + \sqrt{4+5x} &\Rightarrow y - 3 = \sqrt{4+5x} \\ &\Rightarrow (y-3)^2 = 4 + 5x \\ &\Rightarrow \frac{(y-3)^2 - 4}{5} = x \end{aligned}$$

   Then swap x and y we get the inverse function

$$y = \frac{(x-3)^2 - 4}{5}$$

   i.e.

$$f^{-1}(x) = \frac{(x-3)^2 - 4}{5}.$$

   The domain of the inverse function is the range of the original function; which is $[3, +\infty)$.

   Remark: Some of you find the domain of the expression obtained and conclude that the domain of f^{-1} is $\mathbb{R}$. This is incorrect!

2. (a) See note or textbook.

   (b) (For future reference, I change the original statement of this question to more precise one.) By part (a), we know that

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))} = \frac{1}{f'(1)}$$

   Note that $f^{-1}(a) = 1$ because $f(1) = a$. So it remains to compute $f'(1)$.

   By algebraic property of logarithm, one has

$$\ln\left(\frac{x+1}{1 - \sin(x)}\right) = \ln(x+1) - \ln(1 - \sin(x))$$

   Note that the formula works because $1 - \sin(x) \geq 0$. Thus,

$$\begin{aligned} f'(x) &= \ln(x+1)' - \ln(1 - \sin(x))' \\ &= \frac{1}{x+1} - \frac{-\cos(x)}{1 - \sin(x)} \\ &= \frac{1}{x+1} + \frac{\cos(x)}{1 - \sin(x)} \end{aligned}$$

and so

$$(f^{-1})'(a) = \frac{1}{f'(1)} = \frac{1}{\frac{1}{2} + \frac{\cos(1)}{1-\sin(1)}}$$

(c) In class we have seen that the tangent line is given by

$$\begin{aligned} y &= (f^{-1})'(a)(x-a) + f^{-1}(a) \\ &= \frac{1}{\frac{1}{2} + \frac{\cos(1)}{1-\sin(1)}} \left[x - \ln\left(\frac{2}{1-\sin(1)}\right) \right] + 1 \end{aligned}$$

3.

$$\begin{aligned} \int_1^e \frac{3x^2+2x+1}{x} dx &= \int_1^e \left(3x + 2 + \frac{1}{x}\right) dx \\ &= \frac{3x^2}{2} + 2x + \ln|x| \Big]_1^e \\ &= \left(\frac{3e^2}{2} + 2e + \ln|e|\right) - \left(\frac{3\cdot 1^2}{2} + 2\cdot 1 + \ln|1|\right) \\ &= \frac{3e^2}{2} + 2e + 1 - \frac{3}{2} - 2 \\ &= \frac{3e^2}{2} + 2e - \frac{5}{2} \end{aligned}$$

4. (a) Let $y = \tan^{-1}(x)$. By definition, y is the (one and only) real number in $(-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\tan(y) = x$; equivalently

$$\frac{\sin(y)}{\cos(y)} = x.$$

Our goal is to write $y = \sin^{-1}(f(x))$ for some appropriate function $f(x)$; or equivalently, expressing $\sin(y) = f(x)$. Squaring both sides, the above equation implies

$$x^2 = \frac{\sin^2(y)}{\cos^2(y)} \qquad \Rightarrow \qquad 1 + x^2 = \frac{\cos^2(y) + \sin^2(y)}{\cos^2(y)} = \frac{1}{\cos^2(y)}.$$

Cross multiply-divide, we obtain

$$\cos^2(y) = \frac{1}{1+x^2} \qquad \Rightarrow \qquad \sin^2(y) = 1 - \cos^2(y) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2}$$

and we have

$$\sin(y) = \pm\sqrt{\frac{x^2}{1+x^2}} = \pm\frac{x}{\sqrt{1+x^2}}$$

Observe that any angle $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\cos(y) \geq 0$ and so $\sin(y)$ and $\tan(y)$ has the same sign i.e. they are either both positive or both negative or both zero. Thus, we must have

$$\sin(y) = \frac{x}{\sqrt{1+x^2}}.$$

From this identity, we see that

$$y = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)$$

and we have found our desired expression:

$$\tan^{-1}(x) = \sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right).$$

Remark: Many of you found a shorter way to find out the expression using Pythagorean theorem (draw a right triangle with two sides 1 and x). The above method is the formal way to solve the problem (since x might be negative).

(b) Recall that $(\sin^{-1})'(x) = \frac{1}{\sqrt{1-x^2}}$. By chain rule:

$$\begin{aligned}(\tan^{-1})'(x) &= (\sin^{-1})'\left(\frac{x}{\sqrt{1+x^2}}\right) \cdot \left(\frac{x}{\sqrt{1+x^2}}\right)' \\ &= \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{1+x^2}}\right)^2}} \cdot \left(\frac{x}{\sqrt{1+x^2}}\right)' \\ &= \frac{1}{\sqrt{1-\frac{x^2}{1+x^2}}} \cdot \left(\frac{x}{\sqrt{1+x^2}}\right)' \\ &= \frac{1}{\sqrt{\frac{(1+x^2)-x^2}{1+x^2}}} \cdot \left(\frac{x}{\sqrt{1+x^2}}\right)' \\ &= \frac{1}{\sqrt{\frac{1}{1+x^2}}} \cdot \left(\frac{x}{\sqrt{1+x^2}}\right)' \\ &= \sqrt{1+x^2} \cdot \left(\frac{x}{\sqrt{1+x^2}}\right)'\end{aligned}$$

By quotient rule:

$$\begin{aligned}\left(\frac{x}{\sqrt{1+x^2}}\right)' &= \frac{\sqrt{1+x^2} - x(\sqrt{1+x^2})'}{(\sqrt{1+x^2})^2} \\ &= \frac{\sqrt{1+x^2} - x\frac{2x}{2\sqrt{1+x^2}}}{1+x^2} \qquad \text{where } (\sqrt{1+x^2})' = \frac{2x}{2\sqrt{1+x^2}} \text{ by chain rule} \\ &= \frac{(1+x^2)-x^2}{(1+x^2)\sqrt{1+x^2}} \\ &= \frac{1}{(1+x^2)\sqrt{1+x^2}}\end{aligned}$$

So we have

$$\begin{aligned}(\tan^{-1})'(x) &= \sqrt{1+x^2} \cdot \frac{1}{(1+x^2)\sqrt{1+x^2}} \\ &= \frac{1}{1+x^2}\end{aligned}$$

(c) See textbook for the derivation of

$$(\tan^{-1})'(x) = \frac{1}{1+x^2}$$

5. (a) Lecture notes or textbook.

   Remark: Many of you know the purpose of the rule but not the formal statement. I have stressed multiple times in class that it is IMPORTANT TO REMEMBER THE RULE so that you won't misuse it. The key assumption people forgot is $g'(x) \neq 0$ in a neighborhood of a.

   (b) The limit is of the indeterminate form 0^0 so we use the standard trick in class, write the expression in term of exponential

$$\begin{aligned}\tan(8x^2)^x &= \exp(x\ln(\tan(8x^2))) \\ &= \exp\left(x\ln\left(\frac{\sin(8x^2)}{\cos(8x^2)}\right)\right) \\ &= \exp[x(\ln(\sin(8x^2)) - \ln(\cos(8x^2)))]\end{aligned}$$

and since exp is continuous,

$$\lim_{x\to0^+}\tan(8x^2)^x = \exp\left(\lim_{x\to0^+} x[\ln(\sin(8x^2)) - \ln(\cos(8x^2))]\right).$$

Note that as $x \to 0^+$, we have $\sin(8x^2) \to 0$ and $\cos(8x^2) \to 1$ and so $\ln(\sin(8x^2)) \to -\infty$ and $\ln(\cos(8x^2)) \to 0$. By property of limit,

$$\begin{aligned}\lim_{x\to0^+} x[\ln(\sin(8x^2)) - \ln(\cos(8x^2))] &= \lim_{x\to0^+} x\ln(\sin(8x^2)) - \underbrace{\lim_{x\to0^+} x\ln(\cos(8x^2))}_{0\cdot0=0} \\ &= \lim_{x\to0^+} x\ln(\sin(8x^2))\end{aligned}$$

The last limit is of the form $0\cdot(-\infty)$ so we turn the expression into quotient and use L'Hospital's rule:

$$\begin{aligned}\lim_{x\to0^+} x\ln(\sin(8x^2)) &= \lim_{x\to0^+}\frac{\ln(\sin(8x^2))}{1/x} \\ &= \lim_{x\to0^+}\frac{\frac{1}{\sin(8x^2)}\cos(8x^2)16x}{-1/x^2} \\ &= \lim_{x\to0^+}\frac{-\cos(8x^2)16x^3}{\sin(8x^2)} \\ &= \lim_{x\to0^+}(-2x)\frac{8x^2}{\sin(8x^2)} && \text{since } \lim_{x\to0^+}\cos(8x^2) = 1 \\ &= 0 && \text{since } \lim_{x\to0^+}\frac{8x^2}{\sin(8x^2)} = 1\end{aligned}$$

It follows that

$$\lim_{x\to0^+} x[\ln(\sin(8x^2)) - \ln(\cos(8x^2))] = 0$$

and so

$$\lim_{x\to0^+}(\tan(8x^2))^x = \exp(0) = 1.$$