BASIC MATHEMATICAL NOTATIONS

ABSTRACT. This notes is to summarize the basic knowledge about mathematical notations, mathematical expressions and substitution for variable in expressions. Let's forget everything you learn about real numbers temporarily.

1. AMBIGUITY IN THE ORDER OF COMPUTATION

In the beginning, the arithmetic operations of real numbers, namely

1. addition $+$,
2. multiplication (usually denoted by $\times$ in elementary school, by a dot $\cdot$ in college due to the ubiquitous use of x as variable and even omitted entirely in algebraic expressions),
3. subtraction $-$, and
4. division (usually denoted by $\div$ in elementary school or by a slash symbol $/$ and by a fraction $\frac{\bullet}{\bullet}$ form in college),

are *binary operations*: Given **TWO** numbers, I can add, subtract, multiply, divide them. Given only that, it **DOES NOT** make sense to write things like

$$1 + 2 + 3$$

because there is **ambiguity** when evaluating the expression: Should I add $1 + 2$ first, then add 3 to the result? Or should I add $2 + 3$ first, then add 1 to the result? Likewise, expression like

$$1 + 2 \cdot 3$$

has ambiguity: On the one hand, adding $1 + 2$ first and then multiply the number you got with 3, you get 9; whereas if you multiply 2 and 3 first, you get 6 and then adding 1 to it get you 7.

Think of the situation this way: If one day we meet aliens in a galaxy far far away, we come to learn of some operator used alien $\diamond$ and we know they operate in the following way:

$$\bullet \diamond \Delta = \Delta \qquad\qquad \bullet \diamond \bullet = \bullet$$
$$\Delta \diamond \Delta = \bullet \qquad\qquad \Delta \diamond \bullet = \bullet$$

(The first one "bullet/black circle diamonds triangle is triangle" reads just like the elementary "one plus two is three" $1 + 2 = 3$.) Now think about the alien hands you this:

$$\Delta \diamond \Delta \diamond \Delta$$

What should you get as a result? If you do the first operation first, you get $\bullet$ and then diamonds $\diamond$ it with Δ give you Δ as the final result. On the other hand, doing the second $\diamond$ first gives us $\bullet$ whence $\Delta \diamond \bullet$ gives us $\bullet$ as the answer.

$$\underbrace{\underbrace{\Delta \diamond \Delta}_{\bullet} \diamond \Delta}_{\Delta} \qquad\qquad \underbrace{\Delta \diamond \underbrace{\Delta \diamond \Delta}_{\bullet}}_{\bullet}$$

Exercise 1. *Study the ambiguity of order of evaluation in the expressions:*

$$1 + 2 \div 3$$
$$1 - 2 - 3$$
$$1 + 3 \cdot 4 - 7$$
$$1 + 3 \cdot 4 - 7 \div 5$$
$$2 \cdot 9 + 3 - 1 \cdot 4 - 7 \div 5$$

2. Disambiguation Expressions with Parentheses

One way to eliminate the ambiguity of order of computation is to **add parentheses**[1]; writing

$$1 + (2 + 3)$$

indicating that you should do $2 + 3$ first, and then add 1 to the result. Likewise, we write

$$1 + (2 \cdot 3) \qquad \text{or} \qquad (1 + 2) \cdot 3$$

indicate clearly whether you should do addition first or multiplication first.

Note: An expression is called well-formed if there is a balance in opening and closing parentheses. Expressions like

$$(1 + 2$$

or

$$)1 + 2$$

[1] In principal, anything that helps indicating the order works. For example, adding under braces like $1 + \underbrace{2 + 3}$ or boxing $1 + \boxed{2 + 3}$ fulfill the same purpose. But parentheses is the most convenient when it comes to writing. By the way, all kind of parentheses mean the same thing. You can use [ ] or { } in place of ( ).

are not well-formed: the former misses a closing parenthesis where the latter has an orphan closing parenthesis (one without an opening parentheses).

Exercise 2. *Evaluate the expression*

$$1 - \sqrt{3} \cdot 5 \div 9 + 2$$

in different orders to see the difference.

As a remark, parentheses are also used to distinguish between numbers and multiplication. For example, writing 2(3) means multiply 2 and 3 (as mentioned the multiplication sign is typically omitted in college) where as writing 23 denotes a single number, twenty three.

3. Mathematical convention to add missing parentheses

It would be bothersome to put in parentheses all the time. For instance, think of having to write

$$((((1 + 2) + 3) + 4) + 5)$$

and

$$(((x^3 + x^2) + x) + 1)$$

over and over again. Thus, mathematicians come up with a **default interpretation** (a convention, the mathematical world version of ISO, if you will) for expressions with missing parentheses.

In simplest terms, the convention says that multiplication and division has *higher precedence* than addition and subtraction: When there is a lack of parentheses,

- **Addition and subtraction has same precedence.**
- **Multiplication and division has the same precedence; and that precedence is higher than the one of addition/subtraction.**
- **Operation with the same precedence evaluates in writing order (left to right).** E.g. $1 + 2 + 3$ should be interpreted as $(1 + 2) + 3$. Likewise,

$$\begin{aligned} 1 - 2 + 3 &\quad \text{means} \quad (1 - 2) + 3 = 2 \\ 1 \cdot 2 \div 3 &\quad \text{means} \quad (1 \cdot 2) \div 3 = \frac{2}{3} \\ \sqrt{2} + 5 - 7 &\quad \text{means} \quad (\sqrt{2} + 5) - 7 \approx -0.58578644 \\ 1 + 2 + 3 + 4 + 5 &\quad \text{means} \quad (((1 + 2) + 3) + 4) + 5 \end{aligned}$$

- **Operation with higher precedence are evaluated first.** E.g.

$$\begin{aligned} 1-2\cdot 3 &\quad \text{means} \quad 1-(2\cdot 3)=-5 \\ 1+3-2\div 3 &\quad \text{means} \quad (1+3)-(2\div 3)=4-6=-2 \\ 1+3\cdot 2\div 3 &\quad \text{means} \quad 1+((3\cdot 2)\div 3)=1+(6\div 3)=1+2=3 \end{aligned}$$

- **The parentheses has highest precedence.** Writing

$$(1-2)\cdot 3$$

forces a (mathematically educated) reader to do subtraction first, then multiply.

Because of these conventions, if you mean differently from the default interpretation, you **MUST** add parentheses. For instance, if you want to indicate that do subtraction first in $1-2\cdot 3$, write $(1-2)\cdot 3 = -3$. Without the parentheses it will be automatically interpreted as $1-(2\cdot 3)=-5$. You may ask why we take this convention; the answer is just for practical purpose: it simplifies the notation effort for most formulas we encountered in practice.

The convention allows one to add default missing parentheses. The steps to do that are

Step 1: Identify first level sub-expressions. Replace them by letters.

Scan the expression from left to right. If you see a number, it is a sub-expression. If it is an operator, skip it. If you see an opening parenthesis, find the corresponding closing one and the part between the two parentheses you found is a sub-expression.

Let take an example:

$$1+(2-(1+3))\cdot(4\div 3-7)$$

Scan it from left to right:

$$\color{red}{1}\color{black}{+(2-(1+3))\cdot(4\div 3-7)}$$

We first encounter 1 which is a number so it is the first subexpression, let us denote it by a letter A and keep track of the fact that $A=1$. Next, we encounter $+$ which is an operator

$$A\color{red}{+}\color{black}{(2-(1+3))\cdot(4\div 3-7)} \qquad A=1$$

so we skip it. Right after $+$, we find an open parenthesis (right before 2,

$$A+\color{red}{(}\color{black}{2-}\color{blue}{(}\color{black}{1+3}\color{blue}{)}\color{red}{)}\color{black}{\cdot(4\div 3-7)} \qquad A=1$$

and the corresponding closing one[2] is the *second* one after 3; so we have found $B = (2 - (1 + 3))$ is the second subexpression. Again, we replace and keep track of B:

$$A + B\cdot(4 \div 3 - 7) \qquad A = 1, B = (2 - (1 + 3))$$

$$A + B \cdot (4 \div 3 - 7) \qquad A = 1, B = (2 - (1 + 3))$$

Continue this game, we skip the $\cdot$ and then identify the closing parentheses to the (before 4 which is the closing one is the one after 7 so $C = (4 \div 3 - 7)$ is the last sub-expression and we got

$$A + B \cdot C \qquad A = 1$$

$$B = (2 - (1 + 3))$$

$$C = (4 \div 3 - 7)$$

at the end of this step. *The letter expression you got must be free of parentheses.* We call $A + B \cdot C$ the abstracted expression, the overall form for the formula.

Step 2: Add parentheses according to the convention.

Replace each sub-expression with a letter, you will see an expression that has no parentheses. For instance, we now found that the expression in the above example becomes

$$A + B \cdot C$$

which we easily add parentheses according to the convention: $\cdot$ has higher precedence than $+$, so we add parentheses around $B \cdot C$:

$$A + (B \cdot C)$$

Step 3: For each sub-expression found in Step 1, if there is missing parentheses in that sub-expression, temporarily remove outer parentheses if there is, and then repeat the process on that sub-expression. After you process all sub-expressions, add back one pair of parentheses if there was some originally and copy back to the form you get in Step 2.

In the above example, we do not have to do anything for the sub-expressions A and B since they no longer have any ambiguity. As for

$$C = (4 \div 3 - 7)$$

we first temporarily remove the parentheses to get

$$4 \div 3 - 7$$

[2] How to find the closing one? The idea is that parentheses come in pair. So the closing one must be the one so that between the two parentheses, there is the same number of open and closing parentheses.

which you easily add the missing parentheses according to the convention:

$$(4 \div 3) - 7$$

Then we add back the parentheses we stripped from C:

$$((4 \div 3) - 7)$$

and this is the fully parenthesized form for C. Now, assemble our result: we have

$$A = 1 \qquad B = (2 - (1 + 3)) \qquad C = ((4 \div 3) - 7)$$

which we substitute to what we found in Step 2 i.e.

$$A + (B \cdot C)$$

we should get

$$1 + ((2 - (1 + 3)) \cdot ((4 \div 3) - 7))$$

and this is the fully parenthesized expressions: No ambiguity in order of computation. (The colored parentheses are the one we added after this process: The red are due to Step 2 and the blue are due to Step 3.)

Exercise 3. *Add parentheses to the following expressions according to mathematicians' conventions and evaluate the result:*

- $1 - (7 + 5) \cdot 8 + 7 \cdot \sqrt{2} - 8$
- $1 \cdot 7 \div 5 \cdot 8 \div \sqrt{2}$
- $100 + 17 \div \sqrt{2} - 9$

Make 100 random expressions (or find them from the textbook) and add parentheses yourself.

Notes:

- In college, the usage of fraction form eliminates part of ambiguity with division. For example, using

$$\frac{1+2}{3}$$

already means $(1 + 2) \div 3 = 1$ where as

$$1 + \frac{2}{3}$$

means $1 + (2 \div 3) = \frac{5}{3}$. Likewise, square root symbols naturally do not have ambiguity.

- At this stage, the convention **DOES NOT** allow you to remove parentheses from
$$1 + (2 + 3)$$
If you remove it, $1 + 2 + 3$ implicitly means $(1 + 2) + 3$ which is not identical to the above expressions. It **REQUIRES** the associativity of addition of real numbers to remove that parentheses.
There are number systems that DO NOT have associativity for multiplication[3].
- These conventions apply not only to numbers but algebraic expressions as well. For instance,
$$x^2 + x + 1$$
means
$$((x^2) + x) + 1$$
by default. The parentheses around x^2 is because $x^2 = x \cdot x$ is an implicit multiplication, if you write it in full, it should be $((x \cdot x) + x) + 1$! Likewise,
$$y(3y^2 + x - 1) + 5xy$$
implicitly means
$$y((((3y)y) + x) - 1) + (5x)y$$

Exercise 4. *Remove parentheses (if possible) from these expression so that it won't change the meaning due to the default interpretation of the convention:*

- $(1 - (7 + 5) \cdot (8 \div \sqrt{2}))$
- $(1 \cdot (7 \div 5)) \cdot (8 \div \sqrt{2})$
- $100 + 17 \div (\sqrt{2} - 9)$

4. Substitution

I found many of you have problems with substituting (or instantiating) numbers (or expressions) E for a variable x in an expression F. The proper way to do it is to **put parentheses** around E first, then replace all instances of x with that. In short replace all occurrences of x by (E), not E.

[3]If you do physics, there is a chance you will encounter the octanion which is an extension of real numbers, like complex number, but is much more complicated.

For example, substitute $2+\sqrt{2}$ for x in the expression x^2+x+1: Put parentheses $(2+\sqrt{2})$ and then replace all occurrences of x by it, we get

$$(2+\sqrt{2})^2+(2+\sqrt{2})+1$$

as the result. If you don't put parentheses, you will get

$$2+\sqrt{2}^2+2+\sqrt{2}+1$$

which has completely different meaning.

Likewise, substitute the expression $\frac{x^2}{x+1}+2$ for x in the expression $\frac{x^3}{\sqrt{x+5}-1}$, you should get

$$\frac{\left(\frac{x^2}{x+1}+2\right)^3}{\sqrt{\left(\frac{x^2}{x+1}+2\right)+5}-1}$$

as the result, not

$$\frac{\frac{x^2}{x+1}+2^3}{\sqrt{\frac{x^2}{x+1}+2+5}-1}.$$

One more time, we have seen distributivity law

$$A(B+C)=AB+AC$$

I have seen the following way of applying the law

$$x(-x+3)=x-x+x3$$

which completely ruins the law. To apply the law, you must substitute appropriate expressions for the variable A,B,C in the law. Here, we want to substitute: x for A; $-x$ for B and 3 for C. The proper way to do it is to add parentheses and then substitute, namely we have

$$(x)((-x)+(3))=(x)(-x)+(x)(3)$$

Then we use the mathematical convention to remove the redundant parentheses: For example, the parentheses around $(x),(-x)$ and (3) are unimportant so left hand side is the same as $x(-x+3)$. For the RHS, again (x) and (3) are not important, we thus get $x(-x)+x3$ which can be simplified to $-x^2+3x$ thanks to *property of real numbers*:

$x3 = 3x$ by commutativity,

$$\begin{aligned}
x(-x) &= (-x)x && \text{commutativity}\\
&= ((-1)x)x && \text{definition of } -x\\
&= (-1)(xx) && \text{associativity}\\
&= (-1)x^2 && \text{definition of } x^2\\
&= -x^2 && \text{definition of negation}
\end{aligned}$$

I have seen application of the simplification rule

$$\frac{AB}{AC} = \frac{B}{C}$$

in the following way:

$$\frac{\overbrace{\sqrt{x}}^{A}\ \overbrace{+x}^{B}}{\underbrace{\sqrt{x}}_{A}\ \underbrace{\sqrt{x+1}}_{C}} = \frac{x}{\sqrt{x+1}}$$

or even more bizarre

$$\frac{\overbrace{\sqrt{x}}^{A}\ \overbrace{+x}^{B}}{(\underbrace{\sqrt{x}}_{A}\ \underbrace{+3)\sqrt{x+1}}_{C}} = \frac{x}{3\sqrt{x+1}}$$

You should realize why this is not allowed.

Every rule about simplification of algebraic expression IS NOT ABOUT LITERAL SIMPLIFICATION OF STRINGS OF SYMBOLS; but EXPRESS FACTS ABOUT CERTAIN COMPUTATION.

What the rule $\frac{AB}{AC} = \frac{B}{C}$ screams out to you is the fact that: **If you multiply a quantity by a number and then divide the result by that number, you get what you started with.** In particular, if you multiply the number $\frac{B}{C}$ by A, you get $\frac{AB}{C}$; dividing that by A itself i.e. $\frac{AB}{AC}$ should give you what you started with, namely $\frac{B}{C}$. Hence, the equation $\frac{AB}{AC} = \frac{B}{C}$.

Now think out loud what a certain student did above

$$\frac{\sqrt{x}+x}{\sqrt{x}\sqrt{x+1}} = \frac{x}{\sqrt{x+1}}$$

according to the previous paragraph. The computation on the left hand side is: I **add** $\sqrt{x}$ to x and **divide** it by $\sqrt{x}\sqrt{x+1}$. Clearly, this does not match up with what the rule tell you.

To apply the rule, you MUST perform a **substitution** to the placeholders symbol in the rule. In particular, in the first example above, if you want to apply the rule for $A = \sqrt{x}$, $B = +x$ and $C = \sqrt{x+1}$, you must substitute these into the rule and to substitute require adding parentheses; in this case, the invocation of the rule says

$$\frac{(\sqrt{x})(+x)}{(\sqrt{x})(\sqrt{x+1})} = \frac{(+x)}{(\sqrt{x+1})}$$

Now you have to check if the left hand side matches the expression you are given

$$\frac{(\sqrt{x})(+x)}{(\sqrt{x})(\sqrt{x+1})} \stackrel{??}{=} \frac{\sqrt{x}+x}{\sqrt{x}\sqrt{x+1}}$$

Now, you can remove the parentheses around $\sqrt{x}$ and $\sqrt{x+1}$ **in the denominator** because there is no ambiguity arise: The square root symbol already encodes the information about what you should take square root of, namely x and $x+1$ in this case, and then multiply the results. So the two denominators match. But their numerators DO NOT match

$$(\sqrt{x})(+x) \neq \sqrt{x} + x$$

since implicitly $+x = 0 + x = x$ so $(\sqrt{x})(+x) = (\sqrt{x})x = x\sqrt{x} \neq \sqrt{x} + x$.

Exercise 5. *Find 100 expressions and substitute them with and without adding parentheses. Learn if you get the same value or not.*

For example, substitute $\sqrt{3}+7$ *into* $\frac{x^2}{x+1}$.

Most textbook writer are lazy to explain these delicacy. The "safe" distributivity laws should have been written

$$(A)(B+C) = (A)(B) + (A)(C)$$

and likewise, the safe simplification law

$$\frac{(A)(B)}{(A)(C)} = \frac{B}{C}$$

but most authors assume you know how to properly perform substitution so they do not bother to explain.